

FORMAL CONTACT CATEGORIES

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ABSTRACT. To each oriented surface Σ , we associate a differential graded category $\mathcal{Ko}(\Sigma)$. The homotopy category $Ho(\mathcal{Ko}(\Sigma))$ is a triangulated category which satisfies properties akin to those of the contact categories studied by K. Honda. These categories are also related to the algebraic contact categories of Y. Tian and to the bordered sutured categories of R. Zarev.

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1. INTRODUCTION

The purpose of this paper is to associate a differential graded category $\mathcal{Ko}(\Sigma)$ to each oriented surface Σ . This category is used to study comparison problems between the categories associated to surfaces by Seiberg-Witten-type manifold invariants. For example, we prove that the categories associated to the disk $(D^2, 2n)$ with $2n$ marked points by each theory are equivalent and there is a functorial relationship between the categories associated to a surfaces with boundary when they can be defined.

1.1. The unicity of Floer-type invariants of 3-manifolds. In [34, 33] P. Ozsváth and Z. Szabó introduced invariants of 3-manifolds known as the Heegaard-Floer homologies. Depending upon the setting of a parameter U , there are homology groups: $HF_*^-(M)$, $HF_*^+(M)$, $HF_*^\infty(M)$ which fit into a long exact sequence:

$$\cdots \rightarrow HF_*^-(M) \rightarrow HF_*^\infty(M) \rightarrow HF_*^+(M) \rightarrow \cdots . \quad (1.1)$$

When the parameter $U = 0$, there are simpler invariants $\widehat{HF}_*(M)$. The Heegaard-Floer theory has had a profound effect on the study of 3-manifolds and 4-manifolds [19]. This is in part because it was originally conceived of as a means by which one can obtain information in the Seiberg-Witten invariants [5, 24, 49]. The relationship between the Heegaard-Floer homology theory and the Seiberg-Witten Floer homology was recently articulated by two independent groups of researchers: Ç. Kutluhan, Y.-J. Lee, C. H. Taubes [25] and V. Colin, P. Ghiggini, K. Honda [3]. Both teams built upon the earlier work of C. H. Taubes [40], which identified the Seiberg-Witten Floer homologies $\widehat{HM}^*(M)$ with the Embedded Contact Homology $ECH_*(M)$ due to M. Hutchings [16] and M. Hutchings and C. H. Taubes [17, 18]:

$$\Omega : ECH_*(M) \xrightarrow{\sim} \widehat{HM}^*(M).$$

Using the Embedded Contact Homology as an intermediary, both groups completed the diagram:

$$\begin{array}{ccc} & HF_*^-(M) & \\ & \uparrow & \\ ECH_*(M) & & \\ & \searrow \Omega & \\ & \widehat{HM}_*(M), & \end{array}$$

in a fashion which preserved essential properties of the three homology theories. In particular, the maps defined respect decompositions with respect to $Spin^{\mathbb{C}}$ structures, carry invariants of contact structures to invariants of contact structures, preserve the long exact sequence (1.1) and reductions to the simpler, $U = 0$, theory:

$$\widehat{ECH}_*(M) \cong \widehat{HF}_*(M) \cong \widehat{HM}^*(M). \quad (1.2)$$

Intuitively, each component in the equation above corresponds to a codimension 1 piece of a 4-dimensional topological field theory. It is evident that such a theory satisfies the following properties. In codimension 1, a topological field theory associates a chain complex $C(M)$ to each oriented 3-manifold M . The homology of this chain complex $H_*(C(M))$ is an invariant of the diffeomorphism type of the 3-manifold. In codimension 2, a topological field theory associates a differential graded category $\mathcal{C}(\Sigma)$

to each oriented surface Σ . The derived category $D(\mathcal{C}(\Sigma))$ of right $\mathcal{C}(\Sigma)$ -modules [21, 22] is an invariant of the diffeomorphism type of the surface and reversing the orientation of the surface produces the opposite dg category:

$$\mathcal{C}(\bar{\Sigma}) \cong \mathcal{C}(\Sigma)^{\text{op}}.$$

To each 3-manifold X with boundary $\partial X = \Sigma$, there is a right $\mathcal{C}(\Sigma)$ -module X_* . When a 3-manifold M is split along a surface $M = X \cup_{\Sigma} Y$, the invariant $C(M)$ corresponding to M is quasi-isomorphic to the tensor product,

$$C(M) \xrightarrow{\sim} X_* \otimes_{\mathcal{C}(\Sigma)}^{\mathbb{L}} (Y_*)^{\text{op}},$$

of the modules associated to each piece. If the identifications made by Equation (1.2) result from an equivalence between topological field theories then the codimension 2 extensions of these topological field theories must be equivalent as well.

Question. *Is there an equivalence between codimension 2 extensions of Seiberg-Witten Floer, Heegaard-Floer and Embedded Contact Homology?*

In this paper, we study the simpler question of establishing a relationship between the categories associated to oriented surfaces Σ by Heegaard-Floer theory and Embedded Contact theory. These categories ought to be equivalent.

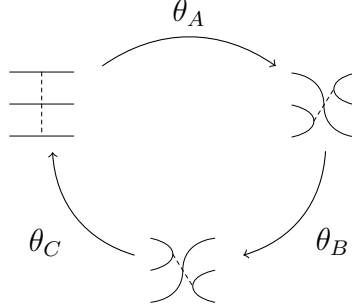
The Heegaard-Floer homology $\widehat{HF}^*(M)$ was extended to surfaces and 3-manifolds with boundary, in the manner described above, by the authors P. Ozsváth, R. Lipshitz and D. Thurston [26]. The theory was further developed by R. Zarev [50, 51]. In particular, when an oriented surface Σ sports a handle decomposition, determined by combinatorial data \mathcal{Z} called an *arc parameterization*, there is a dg category $\mathcal{A}(-\mathcal{Z})$ which is associated to the surface Σ . The Morita homotopy class of the corresponding categories of dg modules are independent of the handle decomposition \mathcal{Z} .

On the contact side, K. Honda has conjectured the existence of a family of triangulated categories $\mathcal{Co}(\Sigma)$ associated to oriented surfaces Σ called *contact categories* [12]. These categories might function as part of a codimension 2 component of the Embedded Contact homology. The morphisms of contact categories are isotopy classes of tight contact structures on a thickened surface $\Sigma \times [0, 1]$. Maps in $\mathcal{Co}(\Sigma)$ are composed by gluing $\Sigma \times [0, 1]$ to $\Sigma \times [0, 1]$ and rescaling. The contact categories $\mathcal{Co}(\Sigma)$ are conjectured to contain distinguished triangles associated to special contact structures called bypass moves. Unfortunately, this construction is not yet available in its full generality. For disks and annuli, algebraic analogues of these categories were introduced and studied by Y. Tian [41, 42].

1.2. Summary of main results. In this paper, we associate a dg category $\mathcal{Ko}(\Sigma)$ to each oriented surface Σ . This category satisfies a universal property which guarantees the existence of a unique map to a dg enhancement of any contact category $\mathcal{Co}(\Sigma)$, when it exists.

Universal property. *If \mathcal{X} is a pretriangulated dg category for which there are choices of maps $\theta : \gamma \rightarrow \gamma'$ corresponding to bypass moves between dividing sets $\gamma, \gamma' \subset \Sigma$ and these maps satisfy four properties:*

- (1) *Bypass moves are cycles.*
- (2) *Trivial bypass moves are equal to identity.*
- (3) *Disjoint bypass moves commute.*
- (4) *Associated to each bypass move is an exact triangle of the form:*



then there is a unique functor $\mathcal{K}o(\Sigma) \rightarrow \mathcal{X}$ in the homotopy category of differential graded categories. See Section 3 for details.

Section 2 contains algebraic background necessary to produce and study the categories $\mathcal{K}o(\Sigma)$. The definition of pre-triangulated hull and a review of Drinfeld-Toën localization construction for dg categories is included. A variation of this localization construction is introduced and related to the standard localization.

Section 3 contains a discussion of surface topology needed for the main construction. The construction of the formal contact categories $\mathcal{K}o(\Sigma)$ follows immediately by combining these topological considerations and the localization construction introduced in Section 2. The remainder of the paper is dedicated to the study of formal contact categories.

In Section 4, we check that the categories satisfy several elementary properties which were outlined by K. Honda. In particular, Corollary 4.13 shows that non-trivial boundary conditions are necessary for Giroux's tightness criterion to be satisfied. Theorem 4.18 shows that when such boundary conditions are present, the triangulated structure allows one to simplify the category by writing dividing sets which do not interact with the boundary in terms of those which do, up to homotopy equivalence. In Section 4.19, formal contact categories $\mathcal{K}o(\Sigma)$ are split into a product of two isomorphic copies of a subcategory $\mathcal{K}o_+(\Sigma)$ called the positive half of the formal contact category.

In Section 5, Theorem 5.3 shows that the mapping class group $\Gamma(\Sigma)$ of Σ acts naturally on the category $\mathcal{K}o(\Sigma)$. Theorem 5.13 shows that when the surface Σ

supports a handle decomposition, determined by an arc parameterization \mathcal{Z} , this produces a collection of generators $\mathfrak{Z}(\mathcal{Z})$ for the category $\mathcal{Ko}(\Sigma)$. After proving the second statement above, Theorem 5.16 uses this result to identify the Grothendieck group $K_0(\mathcal{Ko}_+(\Sigma_{g,1}, 2))$ of the positive half of the formal contact category with the exterior algebra $\Lambda^* H_1(\Sigma_{g,1})$ of the first homology group $H_1(\Sigma_{g,1})$.

The remainder of the paper is dedicated to an investigation of the comparison problem between two codimension 2 extensions: contact categories and Heegaard-Floer categories. The strategy pursued is illustrated by the diagram below:

$$\begin{array}{ccc} \mathcal{Ko}(\Sigma) & \longleftrightarrow & \mathcal{A}(-\mathcal{Z})\text{-mod} \\ & \searrow \quad \swarrow & \\ & \mathcal{Co}(\Sigma) & \end{array}$$

When a reasonable candidate for the geometric contact category $\mathcal{Co}(\Sigma)$ exists, the dashed lines should be taken to be solid.

In Section 6, we study the relationship between three categories associated to the disk $(D^2, 2n)$ with $2n$ points fixed along its boundary. In [41], Y. Tian constructed a candidate \mathcal{Y}_n for $\mathcal{Co}(D^2, 2n)$ and we introduce an arc parameterization \mathcal{M}_n of the disk $(D^2, 2n)$ which gives a dg category $\mathcal{A}(-\mathcal{M}_n)$ associated to the Heegaard-Floer package [50]. The main result of this section is to prove that the three dg categories are Morita equivalent:

$$\mathcal{Co}(D^2, 2n) \cong \mathcal{Ko}_+(D^2, 2n) \cong \mathcal{A}(-\mathcal{M}_n). \quad (1.3)$$

The category $\mathcal{A}(-\mathcal{M}_n)$ is a k -linear category because the differential d is always equal to zero. There are several other instances in which categories with this property can be associated to surfaces. In Section 7, we show that functors from these categories to the homotopy categories of the appropriate formal contact categories can be defined.

Section 8 applies the universal property, discussed above, in a much broader context. The section begins with a discussion of the relationship between the formal contact categories $\mathcal{Ko}(\Sigma)$ and the contact categories $\mathcal{Co}(\Sigma)$. The main theorem leverages the universal property to construct a map:

$$\mathcal{Ko}_+(\Sigma) \rightarrow \mathcal{A}(-\mathcal{Z})\text{-mod}$$

in the homotopy category of dg categories, from the formal contact category associated to Σ to the Heegaard-Floer category associated to Σ , when Σ is parameterized by \mathcal{Z} , for any oriented surface Σ with sufficient boundary conditions.

2. ALGEBRAIC CONSTRUCTIONS

In this section a discussion of localizations follows a review of pretriangulated hulls. Section 2.6 reviews the standard localization procedure for dg categories. Section 2.11 introduces a form of localization which creates formal extensions among objects in a dg category: rather than creating homotopy equivalences amongst objects, this *Postnikov* localization introduces distinguished triangles. In Section 2.20, properties of Postnikov localizations are discussed.

Most of the materials in this section are standard. A review of differential graded categories can be found in [22, 44] or [8, §1]; consult [37, 39, 45] for technical details. The language of model categories is reviewed in the reference [30, §A.2], more details can be found in the references [15, 36].

2.1. Pretriangulated hull. This section contains a brief discussion of pretriangulated hulls of dg categories. The key ideas were introduced in [2, §4]; see also [1, 6].

Definition 2.2. ([6, §2.4]) If \mathcal{C} is a dg category then there exists a dg category $\mathcal{C}^{\text{pretr}}$ called the *pretriangulated hull* of \mathcal{C} . The objects of $\mathcal{C}^{\text{pretr}}$ are one-sided twisted complexes; i.e. formal expressions

$$x = \left(\bigoplus_{i=1}^n x_i[r_i], p \right) \quad \text{such that} \quad dp + p^2 = 0$$

and $n \geq 0, x_i \in \text{Ob}(\mathcal{C}) \cup \{0\}, r_i \in \mathbb{Z}$. The map $p = (p_{i,j})$ is a matrix such that $|p_{i,j}| = 1$ and

$$p_{i,j} = \begin{cases} x_i[r_i] \rightarrow x_j[r_j] & j \geq i \\ 0 & j < i \end{cases}$$

If $x, x' \in \text{Ob}(\mathcal{C}^{\text{pretr}})$ so that $x = (\bigoplus_{i=1}^n x_i[r_i], p)$ and $x' = (\bigoplus_{i=1}^n x'_i[r'_i], p')$ then $\text{Hom}(x, x')$ consists of matrices $f = (f_{i,j})$, $f_{i,j} \in \text{Hom}^{r'_j - r_i}(x_i, x'_j)$, the composition is given by matrix multiplication and the differential $d : \text{Hom}(x, x') \rightarrow \text{Hom}(x, x')$ is determined by the formula:

$$(df)_{i,j} = (df)_{i,j} + (p'f)_{i,j} - (-1)^{|f_{i,j}|}(fp)_{i,j}.$$

Remark 2.3. ([6, §2.4]) If $x, y \in \text{Ob}(\mathcal{C})$ and $f : x \rightarrow y$ is a closed map of degree zero then the cone of f exists in $\mathcal{C}^{\text{pretr}}$ by construction: $C(f) = (x \oplus y[-1], p) \in \text{Ob}(\mathcal{C})$ where $p_{1,2} = f$ and $p_{1,1} = p_{2,1} = p_{2,2} = 0$. The objects in $\mathcal{C}^{\text{pretr}}$ can be obtained by iterated applications of the cone construction.

By construction, the pretriangulated dg category $\mathcal{C}^{\text{pretr}}$ associated to a k -linear category \mathcal{C} factors through its additive closure $\text{Mat}(\mathcal{C})$:

$$\text{Mat}(\mathcal{C})^{\text{pretr}} \cong \mathcal{C}^{\text{pretr}}.$$

The canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{pretr}}$ is fully faithful. A dg category \mathcal{C} is *pretriangulated* when the functor $Ho(\mathcal{C}) \rightarrow Ho(\mathcal{C}^{\text{pretr}})$ induced by inclusion between the associated homotopy categories is an equivalence of categories. The *category of pretriangulated dg categories* will be denoted by $\text{dgc}at_k^{\text{pretr}}$.

The proposition below shows how the pretriangulated hull operation distributes over coproducts of dg categories. It will be used in Theorem 4.5.

Proposition 2.4. *If \mathcal{C}, \mathcal{D} are k -linear then $(\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \cong \mathcal{C}^{\text{pretr}} \amalg \mathcal{D}^{\text{pretr}}$.*

Proof. Since there are no maps between \mathcal{C} and \mathcal{D} , thought of as subcategories of $\mathcal{C} \amalg \mathcal{D}$, a twisted complex $(\bigoplus_{i=1}^n x_i[r_i], p) \in (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}}$ splits into a direct sum of twisted complexes in $\mathcal{C}^{\text{pretr}}$ and $\mathcal{D}^{\text{pretr}}$ respectively. Likewise, matrices $(f_{i,j})$ of maps between twisted complexes in $(\mathcal{C} \amalg \mathcal{D})^{\text{pretr}}$ consist of blocks. It follows that there are functors $\pi_{\mathcal{C}} : (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \rightarrow \mathcal{C}$ and $\pi_{\mathcal{D}} : (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \rightarrow \mathcal{D}$ which satisfy the universal property of the product. \square

The following proposition is well-known, see [1, §1.5].

Proposition 2.5. *The pretriangulated hull $-^{\text{pretr}} : \text{dgc}at_k \rightarrow \text{dgc}at_k^{\text{pretr}}$ is left adjoint to the forgetful functor:*

$$\text{Hom}_{\text{dgc}at_k^{\text{pretr}}}(\mathcal{C}^{\text{pretr}}, \mathcal{D}) \cong \text{Hom}_{\text{dgc}at_k}(\mathcal{C}, \text{Forget}(\mathcal{D})).$$

If $f : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ is a quasi-equivalence then $f^{\text{pretr}} : \mathcal{C}^{\text{pretr}} \rightarrow \mathcal{D}^{\text{pretr}}$ is a quasi-equivalence of dg categories.

The Morita homotopy category Hmo is a refinement of the homotopy category Hqe of dg categories in which derived equivalences are isomorphisms. In Hmo , the homotopy idempotent completion $\mathcal{C}^{\text{perf}}$ of the pretriangulated hull $\mathcal{C}^{\text{pretr}}$ is fibrant replacement, see [37].

2.6. Inverting maps in dg categories. This section contains a brief review of the localization construction for dg categories. Many authors have studied this problem, see [6, 21, 22, 38] and [45, §8.2].

Definition 2.7. The symbol I will be used to denote the dg category freely generated by a cycle $f : 1 \rightarrow 2$ of degree 0 and I' will be used to denote the dg category freely generated by cycles $f : 1 \rightarrow 2$ and $g : 2 \rightarrow 1$ of degree 0.

$$I = 1 \xrightarrow{f} 2 \quad \text{and} \quad I' = 1 \rightleftarrows 2$$

The symbol \bar{I} denotes the dg category with a unique degree zero isomorphism $f : 1 \xrightarrow{\sim} 2$ with $df = 0$. There are canonical inclusions:

$$\kappa : I \hookrightarrow \bar{I} \quad \text{and} \quad \kappa' : I' \hookrightarrow \bar{I}.$$

These maps are determined by the assignments $\kappa(f) = f$, $\kappa'(f) = f$ and $\kappa'(g) = f^{-1}$.

Definition 2.8. Suppose that \mathcal{C} is a dg category and $R : \coprod_{r \in \mathcal{R}} I \rightarrow \mathcal{C}$ is a dg functor. Then the *localization of \mathcal{C} with respect to R* is a dg functor:

$$P : \mathcal{C} \rightarrow L_R \mathcal{C}$$

which satisfies properties (1) and (2) below.

- (1) The pullback map $P^* : \text{Hom}_{\text{Hqe}}(L_R \mathcal{C}, \mathcal{X}) \rightarrow \text{Hom}_{\text{Hqe}}(\mathcal{C}, \mathcal{X})$ is injective.
- (2) The image of the map P^* consists of maps $f : \mathcal{C} \rightarrow \mathcal{X}$ for which there is a map α making the diagram below commute.

$$\begin{array}{ccc} \coprod_{r \in \mathcal{R}} \text{Ho}(I) & \xrightarrow{\text{Ho}(R^* f)} & \text{Ho}(\mathcal{X}) \\ \text{Ho}(\kappa) \downarrow & \nearrow \alpha & \\ \coprod_{r \in \mathcal{R}} \text{Ho}(\bar{I}) & & \end{array}$$

The image $\text{im}(P^*)$ may be denoted by $\text{Hom}_{\text{Hqe}}^I(\mathcal{C}, \mathcal{X})$.

Corollary 8.8 in [45] shows that for any dg category \mathcal{C} and any functor $R : \coprod_{r \in \mathcal{R}} I \rightarrow \mathcal{C}$, there exists a functor $P : \mathcal{C} \rightarrow L_R \mathcal{C}$ in the homotopy category Hqe of dg categories which satisfies the two properties in Definition 2.8 above. The functor $P : \mathcal{C} \rightarrow L_R \mathcal{C}$ is defined to be the homotopy pushout:

$$\begin{array}{ccc} \coprod_{r \in \mathcal{R}} I & \xrightarrow{R} & \mathcal{C} \\ \kappa \downarrow & & \downarrow P \\ \coprod_{r \in \mathcal{R}} \bar{I} & \longrightarrow & L_R \mathcal{C}. \end{array}$$

When the category \mathcal{C} is cofibrant, this homotopy pushout

$$L_R \mathcal{C} = \coprod_{r \in \mathcal{R}} \bar{I} \coprod_{\substack{\mathbb{L} \\ R}} \mathcal{C}$$

can be computed by replacing the inclusion $\kappa : I \hookrightarrow \bar{I}$ by a well-known cofibration $I \hookrightarrow \tilde{I}$. The dg category \tilde{I} appears in Drinfeld where it is denoted by \mathcal{K} [6, §3.7.1].

Definition 2.9. The category \tilde{I} has two objects: 1 and 2. Its maps are generated by the elements: $f \in \text{Hom}_{\tilde{I}}^0(1, 2)$, $g \in \text{Hom}_{\tilde{I}}^0(2, 1)$, $h_{1,1} \in \text{Hom}_{\tilde{I}}^{-1}(1, 1)$, $h_{2,2} \in \text{Hom}_{\tilde{I}}^{-1}(2, 2)$, $h_{1,2} \in \text{Hom}_{\tilde{I}}^{-2}(1, 2)$:

$$\begin{array}{ccc}
& & f, h_{1,2} \\
& \nearrow & \\
h_{1,1} \hookrightarrow 1 & & 2 \hookrightarrow h_{2,2} \\
& \searrow & \\
& & g
\end{array}$$

The differential is determined by the Leibniz rule together with the equations:

$$df = 0, \quad dg = 0, \quad dh_{1,1} = gf - 1_1, \quad dh_{2,2} = fg - 1_2 \quad \text{and} \quad dh_{1,2} = h_{2,2}f - fh_{1,1}.$$

and the maps are subject to no relations.

Remark 2.10. In Definition 2.8, the category I and the map $\kappa : I \hookrightarrow \bar{I}$ can be replaced by the category I' and the map $\kappa' : I' \hookrightarrow \bar{I}$. Suppose that $R : I \rightarrow \mathcal{C}$ and a candidate $R(f)^{-1}$ for the inverse of the map $R(f)$ already exists in the category \mathcal{C} . Then R can be extended to a functor $R' : I' \rightarrow \mathcal{C}$ so that $R'(f) = R(f)$ and $R'(g) = R(f)^{-1}$ and there is an analogous localization:

$$P : \mathcal{C} \rightarrow L_{R'}\mathcal{C} \quad \text{where} \quad L_{R'}\mathcal{C} = \tilde{I} \prod_{R'}^{\mathbb{L}} \mathcal{C}.$$

2.11. Postnikov localization. A variation of the localization procedure discussed in the previous section is introduced. This *Postnikov localization* introduces distinguished triangles rather than homotopy equivalences. In particular, given a sequence

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$$

of maps S in a dg category \mathcal{C} , there is a dg category $L_S\mathcal{C}$ in which this sequence forms a distinguished triangle. Our exposition follows inspiration from [8, §2.4.1] and [38] closely, but see also [23, 32], [6] and [44, §4.3]. Although dg categories are $\mathbb{Z}/2$ -graded the equivalences discussed below commute with the forgetful functor to the ungraded setting introduced in Section 2.25.

First we introduce a dg category D' which corepresents triangles, see Equation (2.1) and Proposition 2.14. Then Definition 2.16 introduces a dg categories \bar{D} and \tilde{D} which corepresent distinguished triangles. A dg functor $\kappa : D' \hookrightarrow \tilde{D}$ will be used to construct the Postnikov localization in Definition 2.18.

Definition 2.12. The symbol D' will be used to denote the dg category freely generated by cycles: $\theta_{1,2} : 1 \rightarrow 2$, $\theta_{2,3} : 2 \rightarrow 3$ and $\theta_{3,1} : 3 \rightarrow 1$.

$$\begin{array}{ccc}
1 & \xrightarrow{\theta_{1,2}} & 2 \\
& \searrow \theta_{3,1} & \swarrow \theta_{2,3} \\
& 3 &
\end{array}$$

The degrees are determined by $|\theta_{1,2}| = 1, |\theta_{2,3}| = 1$ and $|\theta_{3,1}| = 1$.

Since a dg functor $f : D' \rightarrow \mathcal{C}$ is uniquely determined by where it maps the generators in the definition above, there is a bijection between the set of such functors and (symmetric) triangles in \mathcal{C} .

$$Hom_{\text{dgcat}_k}(D', \mathcal{C}) \xrightarrow{\sim} \{ \text{symmetric triangles in } \mathcal{C} \} \quad (2.1)$$

Definition 2.13. If $f, g : D' \rightarrow \mathcal{C}$ are two triangles in \mathcal{C} then f is isomorphic to g when $Ho(f) \cong Ho(g)$ as objects in the functor category $Hom(Ho(D'), Ho(\mathcal{C}))$.

The proposition below states that in the homotopy category Hqe of dg categories the lefthand side of Equation (2.1) above is equal to isomorphism classes of triangles.

Proposition 2.14. ([8, Prop. 2.4.7]) *For any dg category \mathcal{C} , there is a one-to-one correspondence between homotopy classes of functors $f : D' \rightarrow \mathcal{C}$ and isomorphism classes of triangles in \mathcal{C} :*

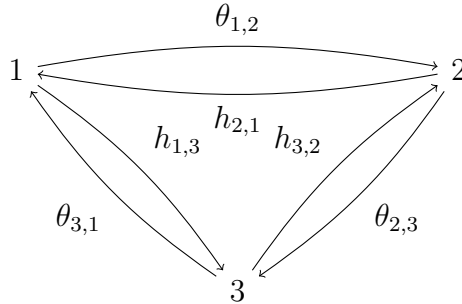
$$Hom_{Hqe}(D', \mathcal{C}) \leftrightarrow \{ \text{symmetric triangles in } \mathcal{C} \} / \text{iso}.$$

Just as isomorphisms are distinguished types of maps, distinguished triangles are distinguished types of triangles. A distinguished triangle is a recipe for constructing one of its objects in terms of the other two.

Definition 2.15. If S is a symmetric triangle $1 \xrightarrow{\theta_{1,2}} 2 \xrightarrow{\theta_{2,3}} 3 \xrightarrow{\theta_{3,1}} 1$ in a dg category \mathcal{C} then S is a *distinguished triangle* if and only if S is isomorphic to the distinguished triangle S' given by $1 \xrightarrow{\theta_{1,2}} 2 \rightarrow C(\theta_{1,2}) \rightarrow 1$ in the homotopy category of $\mathcal{C}^{\text{pretr}}$.

In keeping with Section 2.6, the distinguished property of triangles is formulated as a lifting problem. An innocuous looking dg category \bar{D} which corepresents distinguished triangles and a quasi-equivalent cofibrant replacement $\tilde{D} \xrightarrow{\sim} \bar{D}$ are introduced below.

Definition 2.16. ([8, §2.4.1]) The dg category \bar{D} consists of objects $Ob(\bar{D}) = \{1, 2, 3\}$. The maps are generated by cycles: $\theta_{1,2} : 1 \rightarrow 2$, $\theta_{2,3} : 2 \rightarrow 3$ and $\theta_{3,1} : 3 \rightarrow 1$, and homotopies $h_{2,1} : 2 \rightarrow 1$, $h_{3,2} : 3 \rightarrow 2$ and $h_{1,3} : 1 \rightarrow 3$



with $dh_{2,1} = \theta_{3,1}\theta_{2,3}$, $dh_{3,2} = \theta_{1,2}\theta_{3,1}$ and $dh_{1,3} = \theta_{2,3}\theta_{1,2}$ and the relations:

$$\theta_{2,3}h_{3,2} + h_{1,3}\theta_{3,1} = 1_3, \quad \theta_{1,2}h_{2,1} + h_{3,2}\theta_{2,3} = 1_2, \quad \theta_{3,1}h_{1,3} + h_{2,1}\theta_{1,2} = 1_1.$$

The dg category \tilde{D} consists of objects $Ob(\tilde{D}) = \{1, 2, 3\}$. The maps of degree 1 are $\theta_{i,j} : i \rightarrow j$ are oriented paths and $i \neq j$ in the triangular graph featured in Definition 2.12 above. The differential is zero on paths of length zero or one, when $\theta_{i,i}$ is a cycle, a path of topological degree one, then

$$d\theta_{i,i} = 1_i - \sum_k \theta_{k,i} \theta_{i,k}$$

otherwise $d\theta_{i,j}$ is the sum over compositions of all possible factorizations of the path:

$$d\theta_{i,j} = \sum_k \theta_{k,i} \theta_{j,k}.$$

The projection $p : \tilde{D} \rightarrow \bar{D}$ given by mapping cycles of length 1 to their respective θ -maps is a quasi-equivalence [8, Prop. 2.4.13]. In the other direction, there is an inclusion $\kappa' : D' \hookrightarrow \tilde{D}$ given by sending the θ -maps to their respective length 1 cycles. There is also an inclusion $\kappa' : D' \hookrightarrow \bar{D}$ given by the same formula. A \mathbb{Z} -graded analogue of \tilde{D} is discussed in [23].

The proposition below states that the dg category \tilde{D} corepresents distinguished triangles and satisfies the key properties necessary for the localization construction.

Proposition 2.17. ([8, Prop. 2.4.14])

- (1) *For any dg category \mathcal{C} the set of homotopy classes of dg functors from \tilde{D} to \mathcal{C} is in bijection with the set of isomorphism classes of distinguished triangles in \mathcal{C} :*

$$Hom_{Hqe}(\tilde{D}, \mathcal{C}) = \{1 \xrightarrow{\theta_{1,2}} 2 \xrightarrow{\theta_{2,3}} C(\theta_{1,2}) \rightarrow 1\}/iso.$$

- (2) *The image of the pullback induced by the map κ' appearing in Definition 2.16 coincides with the subset of triangles which are distinguished:*

$$(\kappa')^* : Hom_{Hqe}(\tilde{D}, \mathcal{C}) \rightarrow Hom_{Hqe}(D', \mathcal{C}).$$

- (3) *The set $Hom_{Hqe}(\tilde{D}, \mathcal{C})$ is equal to the set of maps $f \in Hom_{Hqe}(D', \mathcal{C})$ for which there is a map $\alpha : Ho(\tilde{D}) \rightarrow Ho(\mathcal{C})$ such that $Ho(f) = \alpha \circ Ho(\kappa')$.*

We are now ready to discuss a generalization of the localization procedure presented earlier in Section 2.6. Instead of inverting maps in the associated homotopy category, this new operation creates distinguished triangles in the associated homotopy category.

Definition 2.18. Suppose that \mathcal{C} is a dg category and $S : \coprod_{s \in S} D' \rightarrow \mathcal{C}$ is a dg functor. Then the *Postnikov localization of \mathcal{C} with respect to S* is a dg functor:

$$Q : \mathcal{C} \rightarrow L_S \mathcal{C}$$

such that for any dg category \mathcal{X} the following properties are satisfied.

- (1) The pullback map $Q^* : Hom_{Hqe}(L_S \mathcal{C}, \mathcal{X}) \rightarrow Hom_{Hqe}(\mathcal{C}, \mathcal{X})$ is injective and

- (2) The set of maps $Hom_{Hqe}(L_S\mathcal{C}, \mathcal{X})$ in the image of Q^* is equal to the set of maps $f \in Hom_{Hqe}(\mathcal{C}, \mathcal{X})$ such that there is a map α making the diagram below commute.

$$\begin{array}{ccc} \coprod_{s \in S} Ho(D') & \xrightarrow{Ho(f \circ S)} & Ho(\mathcal{X}) \\ \kappa' \downarrow & \nearrow \alpha & \\ \coprod_{s \in S} Ho(\tilde{D}) & & \end{array}$$

The image $im(Q^*)$ may also be denoted by $Hom_{Hqe}^T(\mathcal{C}, \mathcal{X})$.

Recall from above that a functor from $S : D' \rightarrow \mathcal{C}$ is determined by the choice of cycles $f : 1 \rightarrow 2$, $g : 2 \rightarrow 3$ and $h : 3 \rightarrow 1$. The Postnikov localization $L_S\mathcal{C}$ associated to the functor S requires that the sequence:

$$1 \xrightarrow{f} 2 \xrightarrow{g} 3 \xrightarrow{h} 1$$

is a distinguished triangle in the sense of Definition 2.15. The category $L_S\mathcal{C}$ is uniquely determined up to homotopy by the property that a functor $f : \mathcal{C} \rightarrow \mathcal{X}$ factors through $Q : \mathcal{C} \rightarrow L_S\mathcal{C}$ in Hqe when it maps triangles in the image of S to distinguished triangles in the homotopy category $Ho(\mathcal{X})$ of \mathcal{X} .

When \mathcal{C} is a cofibrant dg category, the category $L_S\mathcal{C}$ is a pushout, obtained by gluing a copy of \tilde{D} along the subcategory determined by the image of a functor S . If \mathcal{C} is not cofibrant then $L_S\mathcal{C}$ is a homotopy pushout: the pushout of a cofibrant replacement $\tilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ of \mathcal{C} [30, §A.2.4.4].

The next proposition states that Postnikov localizations always exist.

Proposition 2.19. *For any dg category \mathcal{C} and any collection $S : \coprod_{s \in S} D' \rightarrow \mathcal{C}$, there is a Postnikov localization $Q : \mathcal{C} \rightarrow L_S\mathcal{C}$ in Hqe .*

Proof. It follows from Proposition 2.17 that the functor $\kappa' : D' \rightarrow \tilde{D}$ is a Postnikov localization in the sense of Definition 2.18. Therefore, any coproduct of inclusions: $\coprod_{s \in S} D' \rightarrow \coprod_{s \in S} \tilde{D}$, is an Postnikov localization. For any dg category \mathcal{C} , the localization $Q : \mathcal{C} \rightarrow L_S\mathcal{C}$ is given by the homotopy pushout:

$$\begin{array}{ccc} \coprod_{s \in S} D' & \xrightarrow{S} & \mathcal{C} \\ \downarrow & & \downarrow Q \\ \coprod_{s \in S} \tilde{D} & \longrightarrow & L_S\mathcal{C} \end{array}$$

That $L_S\mathcal{C}$ is a Postnikov localization follows Proposition 2.18 and properties of homotopy pushouts [15]. \square

2.20. Properties of Postnikov localization. In this section we explore properties of the Postnikov localization procedure, establish a relationship between it and the ordinary localization of dg categories, and introduce an analogue of Heller's lemma which facilitates the computation of additive invariants such as the Grothendieck group.

Triangle insertion. The first proposition below assures us that, after having added a triangle, it persists in the pretriangulated hull.

Proposition 2.21. *Suppose that $S : D' \rightarrow \mathcal{C}$, $Q : \mathcal{C} \rightarrow L_S\mathcal{C}$ and $R : L_S\mathcal{C} \rightarrow \mathcal{X}$ is a quasi-fully faithful embedding of the Postnikov localization of \mathcal{C} into a pretriangulated category \mathcal{X} . If $f = RQS(1 \rightarrow 2)$ and $c = RQS(3)$ then c is isomorphic to the cone $C(f)$ of f in the homotopy category of \mathcal{X} .*

$$c \cong C(f) \quad \text{in} \quad Ho(\mathcal{X}).$$

Proof. For the sake of notation, everything to follow takes place inside of the category $Ho(\mathcal{X})$. By TR3 there is a map h in \mathcal{X} which yields a map $(1, 1, h, 1)$ from the triangle $S(1) \rightarrow S(2) \rightarrow S(3) \rightarrow S(1)$ to the triangle $S(1) \rightarrow S(2) \rightarrow C(f) \rightarrow S(1)$. For all $x \in \mathcal{X}$, both triangles determine long exact sequences after applying the functor $Hom(x, -)$. By the Five Lemma $h_* : Hom^*(x, c) \rightarrow Hom^*(x, C(f))$ is an isomorphism. Therefore, Yoneda's lemma implies the result. \square

Decategorification of localizations. For references concerning short exact sequences of dg categories see [22, §4.6].

Lemma 2.22. *Suppose that $S : D' \rightarrow \mathcal{C}$ is a triangle, $\theta_{1,2} = S(1 \rightarrow 2)$ and $c = S(3)$ in a dg category \mathcal{C} . Then S is isomorphic to a distinguished triangle if and only if the double cone complex $K = C(C(\theta_{1,2}) \xrightarrow{\tilde{\theta}_{2,3}} c)$ is contractible where $\tilde{\theta}_{2,3}$ is the extension of the map $\theta_{2,3} : S(2) \rightarrow c$ to the cone $C(\theta_{1,2})$.*

Proof. If S is distinguished then the triangle $S(1) \rightarrow S(2) \rightarrow S(3) \rightarrow S(1)$ is isomorphic to $1 \rightarrow 2 \rightarrow C(\theta_{1,2}) \rightarrow 1$ in the homotopy category via the map $(1, 1, \tilde{\theta}_{2,3}, 1[1])$, so $\tilde{\theta}_{2,3}$ is a homotopy equivalence and $C(\tilde{\theta}_{2,3})$ is contractible. Conversely, $C(\tilde{\theta}_{2,3}) \simeq 0$ implies $\tilde{\theta}_{2,3}$ is a homotopy equivalence and the map above determines an equivalence of triangles. \square

Recall that if $a \in Ob(\mathcal{C})$ then Drinfeld's dg quotient $\mathcal{C}/\langle a \rangle$ can be formed by adding a homotopy h which satisfies $dh = 1_a$ to a cofibrant replacement of \mathcal{C} , see [6]. This

makes the object contractible in the homotopy category of the Drinfeld quotient. (This can be reformulated as a homotopy pushout [38, Thm. 4.0.1].)

The proposition below constructs a short exact sequence of dg categories by relating the Postnikov localization $L_S\mathcal{C}$ of a dg category \mathcal{C} to a Drinfeld quotient $\mathcal{C}/\langle K \rangle$. The subcategory $\langle K \rangle$ is generated by the object K in Lemma 2.22 above.

Proposition 2.23. *Suppose that $S : D' \rightarrow \mathcal{C}$ is a triangle, $f = S(1 \rightarrow 2)$ and $c = S(3)$ in a dg category \mathcal{C} . Then there is a short exact sequence of dg categories:*

$$\langle K \rangle \rightarrow \mathcal{C} \rightarrow L_S(\mathcal{C})$$

in the Morita homotopy category Hmo , where $\langle K \rangle$ is the dg category determined by the cone $K = C(C(f) \rightarrow c)$ of the natural map from the cone on f to c in $\mathcal{C}^{\text{pretr}}$.

Proof. First assume that K is represented by an object in \mathcal{C} . By Definition 2.18, the Postnikov localization $L_S\mathcal{C}$ satisfies the universal property,

$$\text{Hom}_{\text{Hqe}}(L_S\mathcal{C}, \mathcal{X}) \xrightarrow{\sim} \text{Hom}_{\text{Hqe}}^T(\mathcal{C}, \mathcal{X}), \quad (2.2)$$

the set of homotopy classes of functors from $L_S\mathcal{C}$ to any dg category \mathcal{X} is in bijection with the set of homotopy classes of functors $f : \mathcal{C} \rightarrow \mathcal{X}$ which map $\text{im}(S)$ to distinguished triangles in the homotopy categories: $\text{Ho}(f) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{X})$. By the lemma above, the condition that $\text{Ho}(fS) : D' \rightarrow \text{Ho}(\mathcal{X})$ maps to a distinguished triangle is equivalent to the condition that a certain double cone complex K is contractible. If $\tilde{\theta}_{2,3} : C(\theta_{1,2}) \rightarrow 3$ is given by $\tilde{\theta}_{2,3} = (0, \theta_{2,3})$ then set $K = C(\tilde{\theta}_{2,3})$ so that:

$$K = C(\tilde{\theta}_{2,3}) = (1[2] \oplus 2[1] \oplus 3, d_K) \quad \text{where} \quad d_K = \begin{pmatrix} d_1 & \theta_{1,2} & \\ & -d_2 & \theta_{2,3} \\ & & d_3 \end{pmatrix}$$

is contractible in \mathcal{X} . So there is a bijection of sets:

$$\text{Hom}_{\text{Hqe}}^T(\mathcal{C}, \mathcal{X}) \xrightarrow{\sim} \text{Hom}_{\text{Hqe}}^{\langle K \rangle}(\mathcal{C}, \mathcal{X}) \quad (2.3)$$

where $\text{Hom}_{\text{Hqe}}^{\langle K \rangle}(\mathcal{C}, \mathcal{X})$ is the set of maps $f : \mathcal{C} \rightarrow \mathcal{X}$ which send K to a contractible object in \mathcal{X} . Since

$$\text{Hom}_{\text{Hqe}}(\mathcal{C}/\langle K \rangle, \mathcal{X}) \xrightarrow{\sim} \text{Hom}_{\text{Hqe}}^{\langle K \rangle}(\mathcal{C}, \mathcal{X}) \quad (2.4)$$

see [38, Thm. 4.0.1]. The maps in Equations (2.2), (2.3) and (2.4) combine to show that the Postnikov localization satisfies the same universal property as the Drinfeld quotient. Therefore, $\mathcal{C}/\langle K \rangle$ and $L_S\mathcal{C}$ are isomorphic in Hqe . Associated to any such Drinfeld quotient, there is a short exact sequence:

$$\langle K \rangle \hookrightarrow \mathcal{C} \rightarrow \mathcal{C}/\langle K \rangle$$

in the Morita homotopy category \mathbf{Hmo} [38, Rmk. 4.0.2]. Since \mathbf{Hmo} is a quotient of \mathbf{Hqe} , the isomorphism $\mathcal{C}/\langle K \rangle \cong L_S \mathcal{C}$ in \mathbf{Hqe} implies the isomorphism $\mathcal{C}/\langle K \rangle \cong L_S \mathcal{C}$ in \mathbf{Hmo} , and there is a short exact sequence of dg categories:

$$\langle K \rangle \hookrightarrow \mathcal{C} \rightarrow L_S \mathcal{C}.$$

Now suppose that K is *not* representable by object in \mathcal{C} . In the Morita homotopy category \mathbf{Hmo} , the fibrant replacement $\mathcal{C}^{\text{perf}}$ of \mathcal{C} is the category of perfect modules over \mathcal{C} : an idempotent completion of the pretriangulated hull. The object K is representable in $\mathcal{C}^{\text{perf}}$, (see Remark 2.3), and so, by the argument above, there is a short exact sequence:

$$\langle K \rangle \rightarrow \mathcal{C}^{\text{perf}} \rightarrow L_S(\mathcal{C}^{\text{perf}}).$$

In the homotopy category of any model category, every object \mathcal{C} is isomorphic to its fibrant replacement $\beta : \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\text{perf}}$. Since cofibrations in \mathbf{Hmo} and \mathbf{Hqe} are identical, a homotopy pushout in \mathbf{Hqe} is a homotopy pushout in \mathbf{Hmo} . The map β determines an equivalence of pushout diagrams from $\tilde{D} \leftarrow \coprod_s D' \rightarrow \mathcal{C}$ to $\tilde{D} \leftarrow \coprod_s D' \rightarrow \mathcal{C}^{\text{perf}}$ from which it follows that the map $L_S \beta : L_S \mathcal{C} \rightarrow L_S(\mathcal{C}^{\text{perf}})$ is an isomorphism in \mathbf{Hmo} .

There is a commuting diagram extending the righthand side of the short exact sequence in which all of the vertical maps are isomorphisms in \mathbf{Hmo} .

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad} & L_S \mathcal{C} \\ \beta \downarrow & & \downarrow L_S \beta \\ \mathcal{C}^{\text{perf}} & \xrightarrow{\quad} & L_S(\mathcal{C}^{\text{perf}}) \end{array}$$

So there is a short exact sequence: $E \rightarrow \mathcal{C} \rightarrow L_S \mathcal{C}$ where E is a dg category Morita equivalent to $\langle K \rangle$. \square

A short exact sequence of dg categories in \mathbf{Hmo} induces a long exact sequence among additive invariants of dg categories [22, 37]. The corollary below is the first part of the long exact sequence associated to algebraic K-theory.

Corollary 2.24. *Suppose that S , $\langle K \rangle$ and \mathcal{C} are as in the proposition above. Then there is a short exact sequence of abelian groups:*

$$K_0(\langle K \rangle) \rightarrow K_0(\mathcal{C}) \rightarrow K_0(L_S(\mathcal{C})) \rightarrow 0$$

A Postnikov localization as a module. In this section we explain how Postnikov localizations inherit the structure of a module category over $End(\tilde{D})$ in Hmo .

If $\mathcal{C} \cong L_S \mathcal{X}$ is a Postnikov localization of a dg category \mathcal{X} , then the map $\iota : \coprod_{s \in S} \tilde{D} \rightarrow \mathcal{C}$ from the proof of Proposition 2.19 yields a map $\iota^{\text{pretr}} : (\coprod_{s \in S} \tilde{D})^{\text{pretr}} \rightarrow \mathcal{C}^{\text{pretr}}$. Therefore, by Proposition 2.4, there is a map $\hat{\iota}^{\text{pretr}} : \prod_{s \in S} \tilde{D}^{\text{pretr}} \rightarrow \mathcal{C}^{\text{pretr}}$. The pullback of the map $\hat{\iota}^{\text{pretr}}$ along the diagonal map $\Delta_S : \tilde{D}^{\text{pretr}} \rightarrow \prod_{s \in S} \tilde{D}^{\text{pretr}}$ is a functor: $j : \tilde{D}^{\text{pretr}} \rightarrow \mathcal{C}^{\text{pretr}}$. The map j determines an action of $End(\tilde{D}^{\text{pretr}})$ on $\mathcal{C}^{\text{pretr}}$.

$$\begin{array}{ccc} \tilde{D}^{\text{pretr}} & \xrightarrow{j} & \mathcal{C}^{\text{pretr}} \\ \downarrow g & & \downarrow \bar{g} \\ \tilde{D}^{\text{pretr}} & \xrightarrow{j} & \mathcal{C}^{\text{pretr}} \end{array}$$

The universal property in Definition 2.18 gives us a lift \bar{g} of $j \circ g$ for each $g \in End(\tilde{D}^{\text{pretr}})$ and uniqueness of lifts implies that lifts commute with compositions.

2.25. Ungraded dg categories. The main body of the paper will use the trivial grading, a more sophisticated G -grading will be introduced at a later time [4]. Here we require k to be a field of characteristic 2.

There is a category Kom_k^{un} of ungraded chain complexes. In more detail, An *ungraded chain complex* is a k -vector space C and a differential $d_C : C \rightarrow C$ which satisfies $d_C^2 = 0$. A map $f : C \rightarrow D$ of ungraded chain complexes is a map of vector spaces. If $Hom(C, D)$ denotes the vector space of such maps from C to D then there is an associative composition and for each C there is an identity map $1_C : C \rightarrow C$. This determines the category Kom_k^{un} .

The monoidal structure in Kom_k^{un} is the tensor product; the differential is defined by:

$$d_{C \otimes D}(x \otimes y) = d_C x \otimes y + x \otimes d_D y.$$

If $f \in Hom(C, D)$ then the formula $df = fd_C - d_D f$ defines a differential which makes $(Hom(C, D), d)$ an ungraded chain complex and Kom_k^{un} is a category which is enriched over itself.

If $Kom_k^{\mathbb{Z}/2}$ denotes the dg category of $\mathbb{Z}/2$ -graded chain complexes then there is an adjunction

$$\iota : Kom_k^{\text{un}} \leftrightarrow Kom_k^{\mathbb{Z}/2} : \rho$$

in which ι maps (C, d) to the chain complex $(C_n, d_n)_{n \in \mathbb{Z}/2}$ where $C_n = C$ and $d_n = d$ for each $n \in \mathbb{Z}/2$. If $(C_n, d_n)_{n \in \mathbb{Z}/2}$ is a chain complex then $C = \bigoplus_n C_n$ and $d = \sum_n d_n$ determine a forgetful functor $\rho : Kom_k^{\mathbb{Z}/2} \rightarrow Kom_k^{\text{un}}$.

An *ungraded dg category* \mathcal{C} is a category which is enriched over Kom_k^{un} . The adjunction above induces an adjunction between the category $\text{dgc}at_k^{\text{un}}$ of ungraded dg categories and the category $\text{dgc}at_k^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$ -graded categories. This extends to a Quillen adjunction which induces model structures corresponding to Hqe and Hmo on $\text{dgc}at_k^{\text{un}}$, for analogous details see [7, §5.1].

3. FORMAL CONTACT CATEGORIES

In this section, a contact category $\mathcal{Ko}(\Sigma)$ is associated to each oriented surface Σ . The remainder of the paper will assume that k is a field of characteristic 2 and use the trivial grading.

3.1. Bypass moves. In what follows surfaces will always be pointed in the sense defined below.

Definition 3.2. A *pointed surface* Σ is a compact connected surface Σ in which the connected components of the boundary have been ordered and each boundary component $\partial_i \Sigma$ contains a marked point $z_i \in \partial_i \Sigma$:

$$\partial \Sigma = \partial_1 \Sigma \cup \cdots \cup \partial_n \Sigma, \quad z = \{z_1, \dots, z_n\} \quad \text{and} \quad z_i \in \partial_i \Sigma.$$

Every closed surface is canonically pointed.

A pointed oriented surface Σ in which a collection of points $m \subset \partial \Sigma$ satisfy the conditions:

$$m \cap z = \emptyset \quad \text{and} \quad |m| \in 2\mathbb{Z}_+$$

will be denoted by (Σ, m) . We write $m = \cup_i m_i$ where $m_i \subset \partial_i \Sigma$. Often notation will be abused and m will be used to denote both the set m and the cardinality $|m|$.

An orientation on a pointed surface Σ induces an orientation of each boundary component. The points $m_i \subset \partial_i \Sigma$ inherit an ordering by starting from the basepoint $z_i \in \partial_i \Sigma$ and traversing the boundary circle in this direction. Combining the order on each $m_i \subset \partial_i \Sigma$ with the ordering of the boundary components $\{\partial_1 \Sigma, \partial_2 \Sigma, \dots, \partial_n \Sigma\}$ produces a total ordering on the set m .

Recall that an arc γ is properly embedded in a pointed surface when $\partial \gamma \subset \partial \Sigma \setminus z$ and $\text{int}(\gamma) \cap \partial \Sigma = \emptyset$. Arcs γ are required to intersect the boundary transversely.

Definition 3.3. Let Σ be a pointed orientable surface possibly with boundary. Then a properly embedded collection of smooth curves and arcs γ on Σ is a *multicurve*.

If γ is a multicurve on (Σ, m) then we require that the set $\gamma \cap \partial \Sigma$ coincides with the points m chosen on the boundary $\partial \Sigma$.

Definition 3.4. A non-empty multicurve γ is said to be a *dividing set on the surface* Σ when there are disjoint subsurfaces R_+ and R_- of Σ so that

$$\Sigma \setminus \gamma = R_+ \cup R_- \quad \text{and as sets} \quad \gamma = \partial R_+ \setminus \partial \Sigma = \partial R_- \setminus \partial \Sigma.$$

If Σ is a surface with boundary then we require that the intersection number $i(\gamma, \partial \Sigma)$ is a positive even integer. In particular, when Σ has boundary we *require* that $m \geq 2$.

The subsets R_+ and R_- of Σ are the *positive region* and the *negative region* of γ on Σ respectively. These regions may be labelled by $+$ and $-$ signs in illustrations.

If a multicurve γ is a dividing set then for each boundary component $\partial_i \Sigma$, the number of points $\gamma \cap \partial_i \Sigma$ must be even.

Definition 3.5. For any dividing set γ on Σ , there is a *dual dividing set* γ^\vee on Σ that is obtained by exchanging the positive and negative regions.

The *equator* $\ell = \{(x, y) : y = 0\} \subset D^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ of a disk is the line formed by the x -axis in the standard embedding: $D^2 \subset \mathbb{R}^2$. The equator ℓ divides the disk D^2 into two *half-disks*: a bottom B and a top T .

$$D^2 = B \cup T \quad \text{and} \quad B \cap T = \ell$$

The boundary ∂T of the top half-disk T consists of the equator ℓ and the northern hemisphere $\nu \subset \partial D^2$ of the boundary circle:

$$\partial T = \ell \cup \nu.$$

Definition 3.6. Suppose that γ is a dividing set on an oriented surface Σ . Then a *bypass disk on γ* is a smoothly embedded oriented half-disk $(T, \ell) \subset (\Sigma \times [0, 1], \Sigma \times 0)$ which satisfies the following properties:

- (1) The equatorial arc ℓ intersects γ at exactly three points: a, b and c . So that

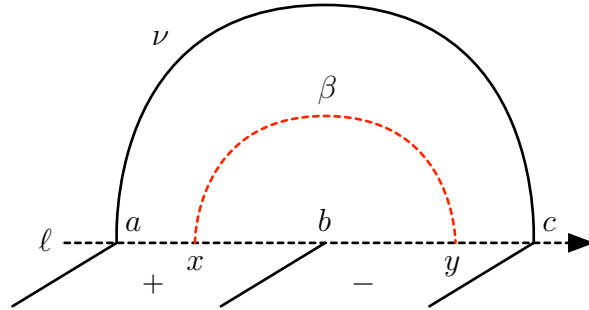
$$\ell = [a, b] \cup [b, c] \quad \text{and} \quad a < b < c.$$

where the order of the points is induced by the orientation.

- (2) The boundary points of the arcs ℓ and ν are the points a and c .

A *dividing set* β of a bypass disk T is a properly embedded arc starting at a point x between a and b and ending at a point y between b and c .

Definition 3.6 above is illustrated below.



The picture above shows a bypass disk T embedded in a thickened surface $\Sigma \times [0, 1]$. The boundary of the half-disk consists of the dashed equatorial arc ℓ and the boundary of the northern hemisphere ν . The dashed red curve β is the dividing set for the bypass disk. The three straight lines at the bottom are part of a dividing set γ on the surface Σ . The labels a, b, c indicate the intersection points of the arc ℓ with the dividing set γ . The orientation of T is determined by fixing the direction of the equator ℓ and using the standard orientation along the normal axis. The equator ℓ is drawn beyond the boundary of T for aesthetic reasons.

Remark 3.7. If $\Sigma \subset (M, \xi)$ is a convex surface in a contact 3-manifold then ξ determines a dividing set γ on Σ . A bypass disk T , embedded into a regular neighborhood of Σ , determines an operation on the dividing set called *bypass attachment* that changes the dividing set and the contact structure in a well-understood way [14]. These operations generate the contact structures on $M = \Sigma \times [0, 1]$ in a sense which has been made precise by K. Honda [13, Lem. 3.10 (Isotopy discretization)].

If Σ is an oriented surface then the space $\Sigma \times [0, 1]$ will be always be oriented by appending the vertical direction to the orientation of Σ .

Definition 3.8. A bypass disk (T, ℓ) in $\Sigma \times [0, 1]$ determines the product orientation on $\Sigma \times [0, 1]$. In more detail, if ℓ represents the direction of the equator and n is the direction of the disk normal to the surface then the three vectors $(\ell, \ell \times n, n)$ determine this orientation of $\Sigma \times [0, 1]$. If the orientation induced by T agrees with that of $\Sigma \times [0, 1]$ then the bypass disk is said to be *orientation preserving*, otherwise it is *orientation reversing*.

Definition 3.9 (Bypass move). Suppose that γ is a dividing set on an oriented surface Σ , T is a bypass disk on γ and $N(T)$ is a regular neighborhood of the half-disk $T \subset \Sigma \times [0, 1]$. The boundary $\partial N(T)$ contains two copies of the half-disk T which we will call *faces*. Each face, being a parallel copy of the half-disk T , contains a collection of points:

$$a < x < b < y < c$$

ordered along an equator ℓ , a dividing set β and a northern hemisphere ν . Moreover, there are three line segments γ_a , γ_b and γ_c from γ , on either side, meeting the points a, b and c respectively. The face in the $\ell \times n$ direction of $T \times \{\frac{1}{2}\} \subset T \times [0, 1]$ is called the *positive face*, the other face is the *negative face*.

There is a dividing set η on the surface $\Sigma' = \partial(\Sigma \cup N(T))$ which is constructed by regluing the curves γ according to the prescription below.

- (1) If T is orientation preserving then on the positive face attach γ_b to the point x of β and attach γ_c to the point y of β and on the negative face attach γ_a to the point x and attach γ_b to the point y .

- (2) If T is not orientation preserving then on the positive face attach γ_a to the point x of β and attach γ_b to the point y of β and on the negative face attach γ_b to the point x and attach γ_c to the point y .
- (3) Attach the curve γ_a on the latter face to the curve γ_c on the former face by an interval that crosses over the $\nu \times [0, 1] \subset \partial N(T)$ boundary component along the diagonal.

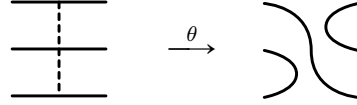
After smoothing the corners, the surface Σ' is diffeomorphic to Σ by a diffeomorphism ψ which is isotopic to identity. If $\gamma' = \psi(\eta)$ then the *bypass move* $\theta : \gamma \rightarrow \gamma'$ is the tuple:

$$\gamma \xrightarrow{\theta} \gamma' = (T, \gamma, \gamma')$$

given by the bypass disk T , the dividing set γ and the curve γ' determined by the operation described above.

Remark 3.10. The definition of bypass move requires a choice of smoothing. We fix one choice and use it consistently. Any two such choices will produce equivalent categories.

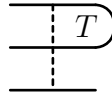
The picture below shows the orientation preserving bypass move defined above. On the lefthand side, the dividing set γ consists of three horizontal lines and the equator ℓ of the bypass disk T is indicated by the vertical line. The rest of the bypass disk T is assumed to come out of the page. The positive and negative regions on the right are determined by the positive and negative regions on the left.



In the contact category, bypass moves are required to be orientation preserving. Since the orientation of a bypass disk T is determined by the direction of the equator, we will always choose orientations which are compatible with the ambient orientation of the surface. So it is not necessary to denote the orientation in most illustrations.

Special types of bypass moves. The two special types of bypass moves isolated below correspond precisely to the relations (1) and (2) in Definition 3.16.

Definition 3.11. A bypass move $\theta : \gamma \rightarrow \gamma'$ is *capped* when either the subset $[a, b]$ or the subset $[b, c]$ of the associated equator ℓ is the equator ρ of an embedded half-disk $(T, T \setminus \rho) \rightarrow (\Sigma, \gamma)$.



Inter-cardinal directions will be used to locate caps. For instance, a bypass featuring a cap T in its northeastern corner is pictured above.

Capped bypass moves are the least interesting bypass moves because, depending upon where the cap is found, a capped bypass must be either nullhomotopic or equal to the identity map in the formal contact category.

Definition 3.12. Two distinct bypass moves $\theta : \gamma \rightarrow \gamma'$ and $\theta' : \gamma \rightarrow \gamma''$ are *disjoint*, up to isotopy with end points fixed in the dividing set, when the equators of their bypass disks have geometric intersection number zero.

If a collection of bypass moves $\{\theta_i\}_{1 \leq i \leq n}$ on a dividing set γ is pairwise disjoint then performing the moves in any order produces the same result: γ' . So the union

$$\Pi_{i=1}^n \theta_i : \gamma \rightarrow \gamma'$$

may be viewed as kind of bypass combo-move.

Isotopy of curves and disks.

Definition 3.13. If γ and γ' are dividing sets on a surface Σ then they are *isotopic*: $\gamma \simeq \gamma'$, when they are isotopic as multicurves on Σ . If Σ is a pointed surface then the isotopy is required to fix the basepoints $z \subset \partial\Sigma$. If (Σ, m) is a surface with points m on each boundary component then the isotopy is required to fix the points at which the dividing sets attach to each boundary component.

Two bypass moves $\theta = (T, \gamma, \gamma')$ and $\theta' = (S, \delta, \delta')$ are *isotopic*: $\theta \simeq \theta'$, when the graph $\gamma \cup \ell$ is isotopic to $\delta \cup \rho$ where ℓ and ρ are equators of T and S respectively.

Remark 3.14. If Σ is realized as a convex surface in the 3-manifold $M = \Sigma \times [0, 1]$ and the dividing sets γ and γ' corresponding to two contact structures ξ and ξ' are isotopic then ξ and ξ' are contactomorphic [14]. Since our motivation is to produce a category in which morphisms behave like contact structures up to contactomorphism, isotopic dividing sets are identified in Definition 3.16 below.

3.15. The contact category.

Definition 3.16. The *pre-formal contact category* $\text{Pre-Ko}(\Sigma)$ is the ungraded k -linear category with objects corresponding to isotopy classes of dividing sets on Σ and maps generated by isotopy classes of orientation preserving bypass moves subject relations below.

- (1) If θ is a capped bypass move then $\theta = 1$ when the cap can be found in the northwest or southeast:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = 1 \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = 1.$$

- (2) If θ and θ' are disjoint bypass moves then the maps that they determine commute:

$$\theta\theta' = \theta \sqcup \theta' = \theta'\theta.$$

The relations above are required for the formal contact category, defined below, to have any bearing on contact geometry, see Remark 3.14 above. In Section 4.9, we will show that the first relation implies that $\theta = 0$ in the associated homotopy category when the corresponding bypass is capped in the northeast or the southwest:

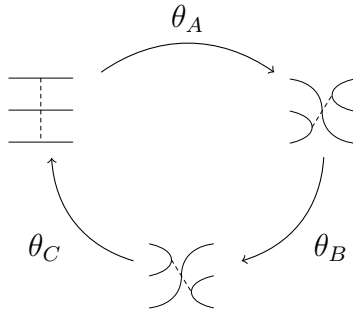
$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = 0.$$

The proposition below is attributed to K. Honda and K. Walker, see [12, 47].

Proposition 3.17. *For each oriented surface Σ and each dividing set γ on Σ , each bypass move θ on γ determines a functor $\tilde{\theta} : D' \rightarrow \text{Pre-Ko}(\Sigma)$.*

Proof. Set $\gamma_A = \gamma$ and $\theta_A = \theta$. By definition, a bypass move $\theta_A = (T_A, \gamma_A, \gamma_B)$ is locally modelled on a bypass disk T_A in $\Sigma \times [0, 1]$ which intersects γ_A in three points. There is a bypass disk T_B on the dividing set γ_B which results from the bypass move θ_A . The disk T_B determines a bypass move $\theta_B = (T_B, \gamma_B, \gamma_C)$ and there is a bypass disk T_C on the dividing set γ_C . The disk T_C determines a bypass move $\theta_C = (T_C, \gamma_C, \gamma_A)$; the result of the bypass T_C is the original dividing set $\gamma = \gamma_A$. These choices are unique up to isotopy. \square

The construction above is illustrated below. Each of the arrows in the diagram is a bypass move. The solid lines represent dividing sets on the surface Σ and the dashed lines represent the equators of bypass disks.



The icon at the source of a given arrow represents a dividing set γ on the surface Σ . The icon at the target of the arrow represents the dividing set obtained by performing the bypass move with equator given by the dashed line in the source.

The proposition above allows us to associate a functor $\tilde{\theta} : D' \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ to each bypass move $\theta : \gamma \rightarrow \gamma'$ between dividing sets on Σ . Composing the coproduct $\coprod_{\theta} \tilde{\theta} : \coprod_{\theta} D' \rightarrow \coprod_{\theta} \text{Pre-}\mathcal{Ko}(\Sigma)$ of all such functors with the fold map $\coprod_{\theta} \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ yields the functor:

$$\Xi : \coprod_{\theta} D' \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma).$$

Definition 3.18. The *formal contact category* $\mathcal{Ko}(\Sigma)$ is the pretriangulated hull of the Postnikov localization of the pre-formal contact category $\text{Pre-}\mathcal{Ko}(\Sigma)$ along the functor Ξ above.

$$\mathcal{Ko}(\Sigma) = L_{\Xi} \text{Pre-}\mathcal{Ko}(\Sigma)^{\text{pretr}}$$

By Proposition 2.21, the bypass triangles introduced by the Postnikov localization remain distinguished triangles in the homotopy category of the hull. The formal contact category $\mathcal{Ko}(\Sigma)$ is the universal pretriangulated category generated by bypass moves, containing bypass triangles and satisfying the relations (1) and (2).

Remark 3.19. A cofibrant-fibrant replacement for $\mathcal{Ko}(\Sigma)$ can be constructed without homotopy pushouts. Note that, before relations (1) and (2) are applied to the pre-formal contact category:

$$\text{Pre-}\mathcal{Ko}(\Sigma) = \text{Pre-Pre-}\mathcal{Ko}(\Sigma) / \langle (1), (2) \rangle,$$

the “pre-pre-formal contact category” is freely generated by bypass moves. Any freely generated category is cofibrant as it can be obtained by a series of pushouts along generating cofibrations in Hqe . One can then adjoin copies of Drinfeld’s category \tilde{I} via pushout and copies of a resolution for the symmetric algebra for each instance of relations (1) and (2) respectively. The result is cofibrant in Hqe , so the homotopy pushout which underlies the Postnikov localization in Definition 3.18 is now an ordinary pushout and the result of this pushout is both cofibrant and fibrant in Hqe . The idempotent completion $L_{\Xi} \text{Pre-}\mathcal{Ko}(\Sigma)^{\text{perf}}$ of $\mathcal{Ko}(\Sigma)$ is cofibrant and fibrant in the Morita category Hmo .

4. ELEMENTARY PROPERTIES OF CONTACT CATEGORIES

In this section many of the properties which should hold for the contact categories [12] are shown to hold for the formal contact categories. The formal contact category associated to a surface decomposes into a product of formal contact categories with fixed Euler invariant. The category with Euler invariant n is equivalent to the category with Euler invariant $-n$. Reversing the orientation of the surface is equivalent to forming the opposite category. A dividing set featuring a homotopically trivial curve is contractible and dividing sets featuring regions which are disconnected from the boundary are shown to be homotopy equivalent to convolutions of dividing sets which are connected to the boundary.

4.1. Decompositions of contact categories. The contact categories $\mathcal{Ko}(\Sigma)$ consist of non-interacting subcategories $\mathcal{Ko}^n(\Sigma, m)$. Each subcategory is determined by fixing some points m on each boundary component and the Euler number $n = \mathfrak{e}(\gamma)$ of the dividing sets γ on Σ .

Euler decomposition. If $(\Sigma \times [0, 1), \xi)$ is a contact 3-manifold and $e(\xi)$ is the Euler class of ξ then the Euler number of ξ is $\mathfrak{e}(\xi) = \langle e(\xi), [\Sigma] \rangle$. This number can be computed from the dividing set $\gamma \subset \Sigma$.

Definition 4.2. If γ is a dividing set on an orientable surface Σ then the *Euler number* $\mathfrak{e}(\gamma)$ of γ is the Euler characteristic of the positive region minus the Euler characteristic of the negative region:

$$\mathfrak{e}(\gamma) = \chi(R_+) - \chi(R_-).$$

The proposition below shows that this is a reasonable thing to consider.

Proposition 4.3. *The Euler number satisfies the following properties:*

(1) *If two dividing sets are isotopic then the corresponding Euler numbers are equal:*

$$\gamma \simeq \gamma' \quad \text{implies that} \quad \mathfrak{e}(\gamma) = \mathfrak{e}(\gamma').$$

(2) *If $\theta : \gamma \rightarrow \gamma'$ is a bypass move then the Euler numbers of γ and γ' must be equal.*

Proof. The first statement follows from the observation that $\gamma \simeq \gamma'$ implies that $R_+ \simeq R'_+$ and $R_- \simeq R'_-$.

The second statement follows from computing each Euler characteristic as a union of the region in which the bypass move is performed and its complement. Suppose that $B \subset \Sigma$ is a small ball containing the bypass moves. If $X_\pm = R_\pm \setminus B$ and $Y_\pm = R_\pm \cap B$ then Y_\pm is homeomorphic to the disjoint union of two disks and

$X_{\pm} \cap Y_{\pm}$ is homeomorphic to the disjoint union of three intervals. See the illustration following Definition 3.9. \square

Remark 4.4. If γ is a dividing set on a surface $(\Sigma_{g,1}, 2)$ of genus g with one boundary component and two points on the boundary then $\chi(R_+ \cap R_-) = 1$ because γ consists of a disjoint union of circles and one interval connecting the two points which are fixed on the boundary. So $2 - 2g = \chi(R_+) + \chi(R_-)$. If $\mathfrak{e}(\gamma) = 2(g - k)$ then $\chi(R_+) = 1 - k$ and $\chi(R_-) = 1 - l$ where $k + l = 2g$ for $0 \leq k \leq 2g$.

Since the pre-formal contact category $\text{Pre-}\mathcal{Ko}(\Sigma, m)$ in Definition 3.16 is generated by bypass moves, the proposition above is equivalent to the statement that the Euler number yields a well-defined map: $\mathfrak{e} : \text{Ob}(\text{Pre-}\mathcal{Ko}(\Sigma, m)) \rightarrow \mathbb{Z}$ which determines a decomposition:

$$\text{Pre-}\mathcal{Ko}(\Sigma, m) \cong \coprod_{n \in \mathbb{Z}} \text{Pre-}\mathcal{Ko}^n(\Sigma, m)$$

in which $\text{Pre-}\mathcal{Ko}^n(\Sigma, m)$ is the full subcategory of $\text{Pre-}\mathcal{Ko}(\Sigma, m)$ such that $\mathfrak{e}(\gamma) = n$ for all $\gamma \in \text{Ob}(\text{Pre-}\mathcal{Ko}^n(\Sigma, m))$. The theorem below shows that this decomposition extends to the formal contact category $\mathcal{Ko}(\Sigma, m)$.

Theorem 4.5. *The formal contact category $\mathcal{Ko}(\Sigma, m)$ splits into a product of categories $\mathcal{Ko}^n(\Sigma, m)$:*

$$\mathcal{Ko}(\Sigma, m) \cong \prod_{n \in \mathbb{Z}} \mathcal{Ko}^n(\Sigma, m)$$

where $\mathcal{Ko}^n(\Sigma, m)$ is the full subcategory of $\mathcal{Ko}(\Sigma, m)$ with objects that satisfy $\mathfrak{e}(\gamma) = n$.

Proof. By the proposition above, $\Xi : \coprod D' \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma, m)$ splits into a union $\Xi = \coprod_n \Xi_n$ where $\Xi_n : \coprod D' \rightarrow \text{Pre-}\mathcal{Ko}^n(\Sigma, m)$ corresponds to the bypass triangles contained in $\text{Pre-}\mathcal{Ko}^n(\Sigma, m)$. The localization functor $Q : \text{Pre-}\mathcal{Ko}(\Sigma, m) \rightarrow L_{\Xi} \text{Pre-}\mathcal{Ko}(\Sigma, m)$ splits into a union of localizations:

$$\text{Pre-}\mathcal{Ko}(\Sigma, m) \cong \coprod_n \text{Pre-}\mathcal{Ko}^n(\Sigma, m) \rightarrow L_{\Xi} \coprod_n \text{Pre-}\mathcal{Ko}^n(\Sigma, m) \cong \coprod_n L_{\Xi_n} \text{Pre-}\mathcal{Ko}^n(\Sigma, m).$$

The theorem follows from Proposition 2.4. \square

4.6. Dualities of contact categories. Two forms of duality are introduced, corresponding to switching the labellings of the regions and the ambient orientation of the surface respectively.

Euler duality. Definition 3.5 introduced an operation $\gamma \mapsto \gamma^\vee$ on dividing sets which exchanged the positive and negative regions: $R_+ \leftrightarrow R_-$. This reverses the sign of the Euler number: $\mathfrak{e}(\gamma^\vee) = -\mathfrak{e}(\gamma)$. Here this operation is extended to an involution

$$-\vee : \mathcal{Ko}(\Sigma, m) \rightarrow \mathcal{Ko}(\Sigma, m)$$

of the formal contact category which exchanges $\mathcal{Ko}^n(\Sigma, m)$ and $\mathcal{Ko}^{-n}(\Sigma, m)$ from Theorem 4.5.

Proposition 4.7. *The Euler duality map on dividing sets: $-\vee : \text{Ob}(\text{Pre-}\mathcal{Ko}^n(\Sigma, m)) \rightarrow \text{Ob}(\text{Pre-}\mathcal{Ko}(\Sigma, m))$ extends to an involution of dg categories:*

$$-\vee : \mathcal{Ko}^n(\Sigma, m) \rightarrow \mathcal{Ko}^{-n}(\Sigma, m) \quad \text{and} \quad (-\vee)^\vee \cong 1.$$

Proof. If γ is a dividing set on Σ then for any bypass move $\theta : \gamma \rightarrow \gamma'$ the positive and negative regions of γ determine positive and negative regions of γ' ; see the illustration after Definition 3.9. Therefore, on the generators θ of $\text{Pre-}\mathcal{Ko}^n(\Sigma, m)$:

$$\theta : \gamma \rightarrow \gamma' \quad \mapsto \quad \theta^\vee : \gamma^\vee \rightarrow \gamma'^\vee.$$

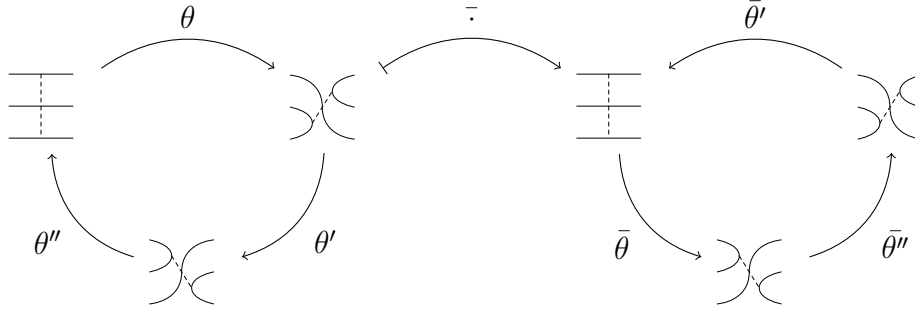
This extends to an involution of $\text{Pre-}\mathcal{Ko}(\Sigma, m)$ which takes triangles to triangles and so descends to a functor: $-\vee : \mathcal{Ko}^n(\Sigma, m) \rightarrow \mathcal{Ko}^{-n}(\Sigma, m)$. The uniqueness of this extension implies the relation $(-\vee)^\vee \cong 1$. The map $-\vee$ is an equivalence as it is its own inverse. \square

Orientation reversal. The formal contact category $\mathcal{Ko}(\bar{\Sigma})$ of a surface with reversed orientation is identified with the opposite formal contact category $\mathcal{Ko}(\Sigma)^{\text{op}}$ of the surface.

Proposition 4.8. *There is an equivalence of formal contact categories,*

$$\mathcal{Ko}^n(\Sigma, m)^{\text{op}} \xrightarrow{\sim} \mathcal{Ko}^n(\bar{\Sigma}, m).$$

Proof. It is a consequence Definition 3.9 that reversing the orientation of the surface is equivalent to reversing the orientation of each bypass half-disk or equator. It suffices to analyze the correspondence between bypass triangles. In the eyeglass-shaped diagram below, reversing the orientation of each bypass disk, $\theta \mapsto \bar{\theta}$ in a triangle fixes the source and changes the sink of each map.



Reversing the arrows on the lefthand side of the diagram produces the bypass triangle for $\mathcal{Ko}^n(\Sigma, m)^{\text{op}}$. The assignment $\gamma \mapsto \gamma$ on objects and $\theta^{\text{op}} \mapsto \bar{\theta}'$ on maps determines a functor $\bar{\cdot} : \text{Pre-}\mathcal{Ko}^n(\Sigma, m)^{\text{op}} \rightarrow \text{Pre-}\mathcal{Ko}^n(\bar{\Sigma}, m)$ because it preserves the cap relations and disjoint unions. Moreover, the relation $\theta^{\text{op}} \mapsto \bar{\theta}'$ implies that $(\theta')^{\text{op}} \mapsto \bar{\theta}''$ and $(\theta'')^{\text{op}} \mapsto \bar{\theta}$ so that triangles are mapped to triangles and the functor $\bar{\cdot}$ descends to a map between formal contact categories. By applying the same construction to the surface after reversing its the orientation again, one obtains an inverse functor and so the functor $\bar{\cdot}$, introduced above, is an isomorphism of formal contact categories. \square

4.9. Relations for overtwisted contact structures. A theorem of E. Giroux [11] states that a contact structure on $\Sigma \times [0, 1]$, when $\Sigma \neq S^2$, is overtwisted if and only if its dividing set contains no homotopically trivial closed curves. When $\Sigma = S^2$, a contact structure is overtwisted if and only if the dividing set contains any two such curves. Corollary 4.13 states that E. Giroux's criteria is satisfied for surfaces with boundary. The surface Σ is assumed to be connected in this section.

The lemma below shows that the local relations can be applied to parts of more complicated dividing sets.

Lemma 4.10. *(Local relations) Suppose that R and Σ are orientable surfaces and $R \subset \Sigma$. Then a distinguished triangle in $\text{Ho}(\mathcal{Ko}(R))$ yields a distinguished triangle in $\text{Ho}(\mathcal{Ko}(\Sigma))$.*

Proof. The embedding $R \subset \Sigma$ determines a functor $i : \text{Pre-}\mathcal{Ko}(R) \hookrightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$. A bypass triangle $\tilde{\theta} : D' \rightarrow \text{Pre-}\mathcal{Ko}(R)$ determines a bypass triangle $D' \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ after composing with i . \square

Definition 4.11. If γ is a dividing set then we write $S^1 \subset \gamma$ when γ contains a homotopically trivial closed curve. All such curves are isotopic when Σ is connected. If γ contains any collection of $n \in \mathbb{Z}_+$ such curves then we write $nS^1 \subset \gamma$.

Proposition 4.12. *The object represented by the dividing set pictured below is contractible.*

$$\begin{array}{c} \bigcirc \\ \smile \end{array} \cong 0$$

Proof. The formal contact category $Ho(Ko(D^2, 2))$ associated to the disk D^2 with two boundary points contains a bypass move with equator indicated by the dashed line below.



All of the objects in the distinguished triangle associated to the bypass move are isotopic and the first relation in Definition 3.16 implies two out of three of the maps are identity. \square

Corollary 4.13. (1) *If Σ is a surface with boundary then for all dividing sets γ on Σ ,*

$$S^1 \subset \gamma \quad \text{implies} \quad \gamma \cong 0 \text{ in } Ho(Ko(\Sigma)).$$

(2) *If Σ is a closed surface then for all dividing sets γ on Σ ,*

$$nS^1 \subset \gamma \text{ and } n \geq 2 \quad \text{implies} \quad \gamma \cong 0 \text{ in } Ho(Ko(\Sigma)).$$

Proof. The proposition above applies to surfaces with boundary as they are required to contain properly embedded arcs. \square

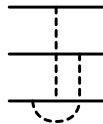
Without further complicating the main construction this corollary appears to be optimal: bypass moves do not imply that $S^1 \cong 0$ in the disk category $Ho(Ko(D^2, 0))$, any such proof would contradict E. Giroux's theorem for $\Sigma = S^2$.

Corollary 4.14. *The relation in Proposition 4.12 above implies that a bypass move is zero in the homotopy category when it is capped in either the northeast or southwest:*

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = 0.$$

Proof. The dividing set γ' resulting from either bypass move $\theta : \gamma \rightarrow \gamma'$ must contain a homotopically trivial curve. So the isomorphism $\gamma' \cong 0$ is obtained by applying Lemma 4.10 and Proposition 4.12. This implies the relation $\theta = 0$ in the homotopy category of the formal contact category. \square

Remark 4.15. Two consecutive bypass moves occurring in a bypass triangle are disjoint:



The second bypass is capped when it is performed before the first, so the commutativity of disjoint bypasses and the corollary above suffice to imply that compositions of consecutive bypass moves must be zero in the homotopy category.

4.16. Dividing sets containing disconnected regions are convolutions. Suppose γ is a dividing set on a surface Σ with boundary and $\Sigma \setminus \gamma$ contains a connected component B which is disjoint from the boundary of Σ . Then we will show that γ is homotopy equivalent to an iterated cone construction on dividing sets which do not contain a region such as B .

Definition 4.17. A multicurve γ on a surface Σ with boundary is *boundary disconnected* when there is a connected component B of $\Sigma \setminus \gamma$ which does not touch the boundary:

$$B \subset \Sigma \setminus \gamma \quad \text{and} \quad B \cap \partial \Sigma = \emptyset$$

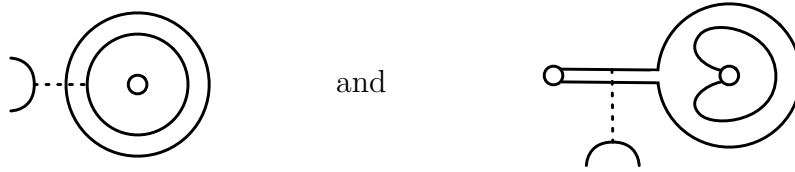
A dividing set γ is *boundary connected* when it is not boundary disconnected.

Theorem 4.18. *In the homotopy category of the formal contact category $Ko(\Sigma, m)$ associated to a surface (Σ, m) with boundary, every boundary disconnected dividing set γ is isomorphic to a convolution of dividing sets γ_i which are boundary connected.*

$$\gamma \cong (\oplus_{i=1}^n \gamma_i, p)$$

Moreover, the components of the twisted differential p are bypass moves.

Proof. If the disconnected region is a disk then γ is isomorphic to zero because $|m| \geq 2$ by Proposition 4.12. If γ is a dividing set on a surface with boundary and $\Sigma \setminus \gamma$ contains an annulus or a punctured torus component then there are bypass moves:



respectively. The first picture above shows two concentric, homotopically non-trivial, circles in the annulus $(S^1 \times [0, 1], 2)$. In the second picture above, the two small circles are identified by folding the page to form a torus with one boundary component $(T^2 \setminus D^2, 2)$. In either case, the triangle associated to the indicated bypass move results in two connected dividing sets.

Now suppose that γ is a disconnected dividing set, then the disconnected region B is an orientable surface with boundary. Any such surface is obtained by attaching 1-handles to the boundary components of a disjoint union of punctured tori $\Sigma_{1,1}$ and annuli $\Sigma_{0,2}$. If B has genus g and $n + 1$ boundary components then B is abstractly homeomorphic to g -copies of $\Sigma_{1,1}$ and n -copies of $\Sigma_{0,2}$ glued together in this fashion.

In particular, there is a 1-handle H which, when cut along an interval I , produces a disjoint union of surfaces with lower genus or number of boundary components. Since Σ has boundary (and so $|m| \geq 2$) there must be at least one boundary connected region so there is a bypass move θ determined by an equator ℓ which begins at this boundary connected region of $\Sigma \setminus \gamma$ and contains the interval I which cuts H . The triangle associated to this bypass move

$$\gamma \xrightarrow{\theta} \gamma' \rightarrow \gamma'' \rightarrow \gamma[1]$$

contains objects γ' and γ'' which must contain disconnected regions, B' and B'' , with lower genus or number of boundary components. Using the bypasses for the annulus and the torus above, the statement of the theorem holds by induction. \square

4.19. The positive half of the contact category. The decomposition of the formal contact category introduced by the proposition below will clarify our discussion later.

Proposition 4.20. *The formal contact category $Ko(\Sigma, m)$ associated to a surface with boundary splits into a product of two pieces:*

$$Ko(\Sigma, m) \cong Ko_+(\Sigma, m) \times Ko_-(\Sigma, m),$$

supported on the dividing sets $\gamma \in Ko(\Sigma, m)$ in which the basepoint $z_1 \in \partial_1 \Sigma$ is contained in a positive or negative region respectively.

Proof. If two dividing sets γ and γ' are isotopic then the signs of the regions containing the basepoint must be equal. If $\theta : \gamma \rightarrow \gamma'$ is a bypass move then it cannot change the sign of the region containing the basepoint z_1 . The rest of the proof follows along the same lines of the proof of Theorem 4.5. \square

By Proposition 4.7, the two pieces found in the decomposition above are equivalent:

$$-\vee : Ko_+^n(\Sigma, m) \xrightarrow{\sim} Ko_-^n(\Sigma, m).$$

In Corollary 5.4, moving the basepoint z_1 to an adjacent region is shown to yield an equivalence $r : Ko_+^n(\Sigma, m) \xrightarrow{\sim} Ko_-^n(\Sigma, m)$. By composing the two maps we obtain an equivalence:

$$Ko_+^n(\Sigma, m) \xrightarrow{\sim} Ko_+^{-n}(\Sigma, m).$$

See also Proposition 6.19.

5. SYMMETRIES AND GENERATORS OF CONTACT CATEGORIES

The mapping class group of the surface Σ is shown to act naturally on the formal contact category $\mathcal{Ko}(\Sigma)$. After introducing arc diagrams and parameterizations of surfaces by arc diagrams, each parameterization of Σ by an arc diagram is shown to yield a system of generators for the formal contact category. A specific choice of generators yields a computation of the Grothendieck group $K_0(\mathcal{Ko}_+(\Sigma_{g,1}, 2))$.

5.1. The mapping class group action. In this section we show that the mapping class group $\Gamma(\Sigma)$ acts naturally on the formal contact category $\mathcal{Ko}(\Sigma)$.

Definition 5.2. Suppose that Σ is an oriented surface. Then the mapping class group $\Gamma(\Sigma)$ is the group of connected components of the group of orientation preserving and boundary fixing diffeomorphisms:

$$\Gamma(\Sigma) = \pi_0 \text{Diff}^+(\Sigma, \partial\Sigma).$$

Recall that an action of a group G on a dg category \mathcal{C} is a homomorphism from G to the group $\text{Aut}(\mathcal{C}) \subset \text{End}_{\text{Hmo}}(\mathcal{C})$ of derived equivalences.

Theorem 5.3. *The mapping class group $\Gamma(\Sigma)$ acts naturally on the formal contact category $\mathcal{Ko}(\Sigma)$.*

Proof. The proof occurs in two steps: first we construct a natural $\Gamma(\Sigma)$ -action on the pre-formal contact category $\text{Pre-}\mathcal{Ko}(\Sigma)$ and second this group action is extended to the formal contact category $\mathcal{Ko}(\Sigma)$.

A diffeomorphism class $g \in \Gamma(\Sigma)$, determines a functor $f_g : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ that is defined by its action on dividing sets and bypass moves. If γ is an isotopy class of dividing set on Σ then there is a unique isotopy class of dividing set $g\gamma$ and if $\theta = (T, \gamma, \gamma')$ is a bypass move then there is a unique bypass disk gT and associated bypass move $g\theta = (gT, g\gamma, g\gamma')$. Since the category $\text{Pre-}\mathcal{Ko}(\Sigma)$ is generated by bypass moves and the assignment $\theta \mapsto g\theta$ preserves disjointness of bypass moves and caps of bypass moves, there is a functor

$$f_g : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma) \quad \text{such that} \quad f_g(\gamma) = g\gamma \text{ and } f_g(\theta) = g\theta.$$

Both the composition law $f_{gg'} = f_g \circ f_{g'}$ and naturality follow directly from the definition. In particular, since the identity diffeomorphism $1 \in \Gamma(\Sigma)$ fixes both dividing sets and bypass moves the functor f_1 is the identity functor $1_{\text{Pre-}\mathcal{Ko}(\Sigma)}$.

Suppose that $f_g : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ is a functor occurring in the construction above. Composing with the localization functor $Q : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)$ from Equation (3.15) yields a functor $\text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)$. By Definition 2.18, the image of $Q^* : \text{Hom}_{\text{Hqe}}(L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma), L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)) \rightarrow \text{Hom}_{\text{Hqe}}(\text{Pre-}\mathcal{Ko}(\Sigma), L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma))$ is the subset of functors $f : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow$

$L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)$ whose restriction to a bypass triangle extends to a distinguished triangle in the localization $L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)$.

If $\tilde{\theta} : D' \rightarrow \text{Pre-}\mathcal{Ko}(\Sigma)$ is the bypass triangle:

$$\gamma \xrightarrow{\theta} \gamma' \xrightarrow{\theta'} \gamma'' \xrightarrow{\theta''} \gamma[1]$$

associated to a bypass move $\theta = (T, \gamma, \gamma')$ on Σ by Proposition 3.17 then $f_g(\theta) = (gT, g\gamma, g\gamma')$ and $f_g(\tilde{\theta})$ corresponds to the bypass triangle:

$$g\gamma \xrightarrow{g\theta} g\gamma' \xrightarrow{g\theta'} g\gamma'' \xrightarrow{g\theta''} g\gamma[1].$$

Since the criteria of Definition 2.18 are satisfied, there is a unique lift of the functor $Q \circ f_g$ to a functor: $\tilde{f}_g : L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow L_{\Xi}\text{Pre-}\mathcal{Ko}(\Sigma)$. By Proposition 2.5, there is an induced functor between the associated pretriangulated hulls:

$$h_g : \mathcal{Ko}(\Sigma) \rightarrow \mathcal{Ko}(\Sigma) \quad \text{where} \quad h_g = \tilde{f}_g^{\text{pretr}}.$$

Uniqueness of the lift and functoriality of $-^{\text{pretr}}$ imply that the stated group action is obtained. \square

The same argument as above allows us to define an automorphism r which moves the first basepoint across the first adjacent boundary point. The corollary below records the existence of this map.

Corollary 5.4. *There is a distinguished automorphism r of $\mathcal{Ko}(\Sigma, m)$ which moves the first basepoint $z_1 \in \partial_1 \Sigma$ on the first boundary component over the nearest boundary point in the direction of the orientation.*

The functor r induces functors $r : \mathcal{Ko}_{\pm}^n(\Sigma, m) \rightarrow \mathcal{Ko}_{\pm}^n(\Sigma, m)$ with respect to the decomposition of $\mathcal{Ko}^n(\Sigma, m)$ found in Proposition 4.20. See also Proposition 6.19.

5.5. Arc diagrams. An arc diagram is a combinatorial way to record a handle decomposition of a surface. The definitions below are due to R. Zarev [50] and constitute generalizations of ideas which were used by R. Lipshitz, P. Ozsváth and D. Thurston [26, §3.2].

Definition 5.6. An *arc diagram* \mathcal{Z} consists of three things:

- (1) an ordered collection $Z = \{\mathcal{Z}_1, \dots, \mathcal{Z}_{\ell}\}$ of ℓ oriented line segments,
- (2) a set $\mathbf{a} = \{a_1, \dots, a_{2k}\}$ of distinct points in the line segments Z and
- (3) a two-to-one function $M : \mathbf{a} \rightarrow \{1, \dots, k\}$ called the *matching*.

In order to apply to any version of the Bordered Heegaard-Floer package, this data is required to be *non-degenerate*: after performing surgery on Z at each 0-sphere $M^{-1}(j)$, for $1 \leq j \leq k$, the resulting 1-manifold has no closed components.

The set of points \mathbf{a} receives a total ordering from the order on the set Z and the orientations of the line segments. The numbers ℓ and k are allowed to be zero. Each arc diagram \mathcal{Z} determines a surface $F(\mathcal{Z})$.

Definition 5.7. The *surface* $F(\mathcal{Z})$ associated to an arc diagram \mathcal{Z} is given by thickening each line segment \mathcal{Z}_i to $\mathcal{Z}_i \times [0, 1]$ for $1 \leq i \leq \ell$ and attaching oriented 1-handles $D^1 \times D^1$ along the normal bundles of the 0-spheres $M^{-1}(j) \times \{0\}$ for $1 \leq j \leq k$. The surface $F(\mathcal{Z})$ is oriented by extending the orientation of the line segment \mathcal{Z}_1 and its positive normal.

Recall that the points m on a pointed oriented surface (Σ, m) are also ordered by the ordering of the boundary components and the order on each boundary component is obtained by starting from each basepoint and following in the direction of the orientation induced on the boundary.

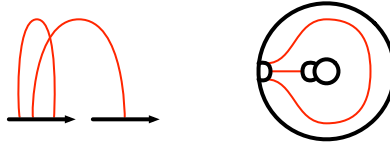
Definition 5.8. Suppose that $m \subset \partial\Sigma$ is the set of *sutures* or points fixed along the boundary of Σ . An *arc parameterization* $(\mathcal{Z}, \varphi_{\mathcal{Z}})$ of a pointed oriented surface (Σ, m) is an arc diagram \mathcal{Z} and a proper orientation preserving diffeomorphism

$$\varphi_{\mathcal{Z}} : (F(\mathcal{Z}), \cup_{i=1}^{\ell} \partial\mathcal{Z}_i) \rightarrow (\Sigma, m)$$

which preserves total order on the points \mathbf{a} and m respectively.

Remark 5.9. The sets m and \mathbf{a} play different roles. The set m may be viewed as a subset $\cup_{i=1}^{\ell} \partial\mathcal{Z}_i \subset \mathbf{a}$. See the examples below.

Example 5.10. The annulus $(S^1 \times [0, 1], (2, 2))$ with two points fixed on each boundary component is parameterized by the arc diagram \mathcal{Z} pictured on the left below.



This picture contains two oriented lines $Z = \{\mathcal{Z}_1, \mathcal{Z}_2\}$ and four points $\mathbf{a} = \{x, x', y, y'\}$ with $\mathcal{Z}_1 = xyx'$ and $\mathcal{Z}_2 = y'$. The matching function $M : \mathbf{a} \rightarrow \{1, 2\}$ is determined by the assignments $M(x) = 1 = M(x')$ and $M(y) = 2 = M(y')$. The picture on the right shows the surface $F(\mathcal{Z})$ associated to \mathcal{Z} .

5.11. Generators from arc diagrams. In this section we show that a parameterization $\mathcal{P} = (\mathcal{Z}, \varphi_{\mathcal{Z}})$ of a pointed oriented surface (Σ, m) determines a canonical collection $\mathfrak{Z}(\mathcal{Z})$ of generators for the associated contact category $\mathcal{Ko}(\Sigma, m)$. This material is motivated by a reading of R. Zarev [51].

Definition 5.12. Suppose that a pointed oriented surface (Σ, m) is parameterized by an arc diagram \mathcal{Z} . Then for each subset $C \subset \{1, \dots, k\}$ of matched pairs, there is an *elementary dividing set*

$$\mathfrak{z}_C = \partial R_C \quad \text{on} \quad \Sigma$$

where $R_C \subset \Sigma$ is the union of a thickening of the core of each 1-handle indexed by C with the collection of thickened oriented arcs $\mathcal{Z}_i \times [0, 1]$. The region R_C is the positive region of \mathfrak{z}_C and its complement $\Sigma \setminus R_C$ is the negative region of \mathfrak{z}_C .

An elementary dividing set may be also be called a *positive elementary dividing set*. The set of positive elementary dividing sets will be denoted by $\mathfrak{z}_+(\mathcal{Z})$. The set of negative elementary dividing sets $\mathfrak{z}_-(\mathcal{Z}) = \mathfrak{z}_+(\mathcal{Z})^\vee$ are obtained by reversing the positive and negative regions. The set of elementary dividing sets is the union

$$\mathfrak{z}(\mathcal{Z}) = \mathfrak{z}_+(\mathcal{Z}) \cup \mathfrak{z}_-(\mathcal{Z}).$$

Theorem 5.13. Suppose that (Σ, m) is a pointed oriented surface with boundary and (Σ, m) is parameterized by an arc diagram \mathcal{Z} . Then the set of elementary dividing sets $\mathfrak{z}(\mathcal{Z})$ generate the contact category $\text{Ko}(\Sigma, m)$: If γ is any dividing set then

$$\gamma \simeq (\oplus_i \mathfrak{z}_{C_i}, p)$$

and the map p consists of bypass moves between elementary dividing sets.

Proof. Suppose that γ is a dividing set on Σ . By Theorem 4.18 we assume that γ is boundary connected. If c_i is the cocore of a 1-handle in $F(\mathcal{Z})$ and $|\gamma \cap c_i| > 2$ then there is a bypass disk with equator parallel to c_i with the property that the associated bypass triangle $\gamma \rightarrow \gamma' \xrightarrow{\theta_B} \gamma'' \rightarrow \gamma[1]$ implies that γ is isomorphic to a cone:

$$\gamma \cong C(\theta_B) \quad \text{such that} \quad |\gamma' \cap c_i|, |\gamma'' \cap c_i| < |\gamma \cap c_i|.$$

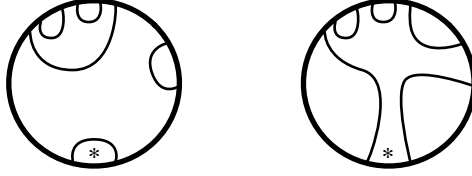
Therefore, we can assume that

$$|\gamma \cap c_i| = 0 \quad \text{or} \quad |\gamma \cap c_i| = 2 \quad \text{for} \quad 1 \leq i \leq k. \quad (5.1)$$

If the intersection number is 0 then the i th 1-handle is *unoccupied* and if the number is 2 then the i th 1-handle is *occupied*. Equation (5.1) is preserved by the remainder of the proof.

After removing the cocores $\Sigma \setminus \{c_1, \dots, c_k\} = \sqcup_{i=1}^\ell D_i^2$ from the surface, one obtains a disjoint union of disks D_i^2 . A dividing set γ which has been reduced in the manner described by the first paragraph must intersect the boundary of each disk D_i^2 at the points where occupied 1-handles are attached and the end points of the oriented line segment $\mathcal{Z}_i \subset \partial D_i^2$.

The two disks pictured below illustrate the next step in the proof.

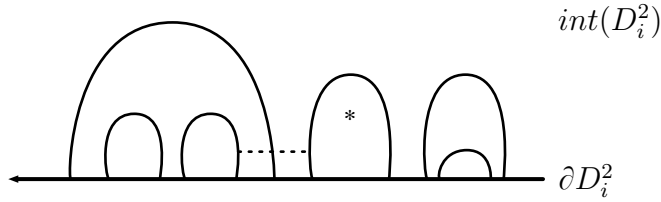


(Here $*$ represents the region containing the oriented line segment \mathcal{Z}_i at its base.) The first disk contains a generic dividing set in D_i^2 and the second contains a dividing set which comes from elementary generators. In the second picture, the region containing $*$ is connected. The rest of the proof explains how to transform the first case into the second case.

More specifically, we show that γ is homotopic to a convolution $(\oplus_k \gamma_k, p)$ such that for each dividing set, γ_k , the signed region of $D_i^2 \setminus \gamma_k$ containing the oriented line \mathcal{Z}_i is connected.

Assume that γ satisfies this property for disks D_j^2 for $0 \leq j < i$. We will simplify γ within the disk D_i^2 without affecting the validity of the statement for the earlier disks.

Traversing the disk D_i^2 , beginning from the region containing \mathcal{Z}_i in the direction of the orientation traces out a finite collection of, potentially nested, regions. Each region is a thickening of a tree. The edges of the trees that do not meet the base region containing the oriented line \mathcal{Z}_i must interface between occupied 1-handles.



The distinguished triangle associated to the bypass move indicated above implies that the picture is homotopic to a cone complex, the first term of which connects the trees together and the second term increases the complexity of the first tree and decreases the complexity of the second. Iterating this procedure on the second tree produces a convolution in which the original trees are connected.

If we continue to connect the trees as we go around the disk then nesting may produce disconnected regions, but these can be removed with Proposition 4.12 and Theorem 4.18. Note that the pattern of occupied 1-handles changes, but this can only decrease the intersection number of a dividing set with a cocore. Isotoping a

bigon through a 1-handle will not affect the the connectedness the dividing set in the earlier disks D_j^2 for $1 \leq j < i$.

We have shown that each dividing set is a convolution of thickenings of chords which connect some subset of 1-handles to each base region, $\mathcal{Z}_i \times [0, 1]$, of each disk containing occupied 1-handles, these are the elementary dividing sets. \square

Corollary 5.14. *When a pointed oriented surface Σ is parameterized by an arc diagram \mathcal{Z} , the positive half of the formal contact category $\mathcal{Ko}_+(\Sigma)$ is generated by the positive elementary dividing sets $\mathfrak{Z}_+(\mathcal{Z})$.*

5.15. Decategorification.

Theorem 5.16. *If $\Sigma_{g,1}$ is the surface of genus g with one boundary component and there are two points fixed on the boundary then the Grothendieck group of the (positive, ungraded) formal contact category is isomorphic to the \mathbb{F}_2 -exterior algebra on the first homology of the surface $\Sigma_{g,1}$:*

$$K_0(\mathcal{Ko}_+(\Sigma_{g,1}, 2)) \xrightarrow{\sim} \Lambda_{\mathbb{F}_2}^{2g+*} H_1(\Sigma_{g,1}; \mathbb{F}_2).$$

Proof. The action $\sigma : \text{Pre-}\mathcal{Ko}_+(\Sigma_{g,1}, 2) \rightarrow \mathcal{Co}(\Sigma_{g,1}, 2)$ of the preformal contact category on the geometric (not necessarily triangulated) contact category discussed in Section 8.1 implies non-triviality of $\text{Pre-}\mathcal{Ko}_+(\Sigma_{g,1}, 2)$. By construction $\text{Pre-}\mathcal{Ko}_+(\Sigma_{g,1}, 2)$ is free subject to relations (1) and (2). Since neither of these relations establish isomorphisms between different objects, the Grothendieck group of $\text{Pre-}\mathcal{Ko}_+(\Sigma_{g,1}, 2)$ is isomorphic to the free \mathbb{F}_2 -module on the set of dividing sets. Proposition 2.23 implies the short exact sequence:

$$K_0(\langle I \rangle) \rightarrow K_0(\text{Pre-}\mathcal{Ko}_+(\Sigma_{g,1}, 2)) \rightarrow K_0(\mathcal{Ko}_+(\Sigma_{g,1}, 2)) \rightarrow 0.$$

Choose the parameterization $\mathcal{Z}_{g,1}$ of $(\Sigma_{g,1}, 2)$ determined by a thickening of the curves representing the standard symplectic basis of $H_1(\Sigma_{g,1}; \mathbb{F}_2)$, see Definition 7.10. The positive elementary dividing sets $\mathfrak{Z}_+(\mathcal{Z}_{g,1})$ of this arc parameterization correspond to the set of subsets of these curves so that there is an isomorphism of vector spaces:

$$\mathbb{F}_2 \langle \mathfrak{Z}_+(\mathcal{Z}_{g,1}) \rangle \xrightarrow{\sim} \Lambda_{\mathbb{F}_2}^* H_1(\Sigma_{g,1}; \mathbb{F}_2).$$

By Corollary 5.14, the Zarev generators $\mathfrak{Z}_+(\mathcal{Z}_{g,1})$ generate $\mathcal{Ko}_+(\Sigma_{g,1}, 2)$, so the Grothendieck group $K_0(\mathcal{Ko}_+(\Sigma_{g,1}, 2))$ is a quotient of $\mathbb{F}_2 \langle \mathfrak{Z}_+(\mathcal{Z}_{g,1}) \rangle$ by the relation:

$$\mathfrak{z} - \mathfrak{z}' + \mathfrak{z}'' = 0 \quad \text{for each triangle} \quad \mathfrak{z} \xrightarrow{\theta} \mathfrak{z}' \rightarrow \mathfrak{z}'' \rightarrow \mathfrak{z}[1], \mathfrak{z}, \mathfrak{z}', \mathfrak{z}'' \in \mathfrak{Z}_+(\mathcal{Z}_{g,1}).$$

With this arc parameterization there are no such bypass triangles: for any bypass $\theta : \mathfrak{z}_C \rightarrow \mathfrak{z}_{C'}$ between elementary dividing sets, the third dividing set γ'' in the associated bypass triangle,

$$\mathfrak{z}_C \xrightarrow{\theta} \mathfrak{z}_{C'} \rightarrow \gamma'' \rightarrow \mathfrak{z}_C[1],$$

is not an elementary dividing set. In order to see that this is so, observe that any bypass move $\theta : \mathfrak{z}_C \rightarrow \mathfrak{z}_{C'}$ corresponds to a choosing of one of the 1-handles α in C and regluing it. First cut α along its cocore as to create two ends A and B . If $\mathfrak{z}_{C'}$ results from gluing the A end to a thickening of the equator ℓ of the bypass then the third term in the bypass triangle associated to θ is obtained by gluing the B end to the equator in the same manner. By construction α from \mathfrak{z}_C corresponds to a basis vector in $H_1(\Sigma_{g,1}; \mathbb{F}_2)$. Since attaching A to the equator gives another generator $\mathfrak{z}_{C'}$ we obtain another basis vector β from $\mathfrak{z}_{C'}$ in $H_1(\Sigma_{g,1}; \mathbb{F}_2)$, but then attaching the B end results in the sum $\alpha + \beta \in H_1(\Sigma_{g,1}; \mathbb{F}_2)$ which is not a basis vector. So the dividing set $\gamma'' \notin \mathfrak{Z}_+(\mathcal{Z}_{g,1})$ is not elementary with respect to the arc parameterization $\mathcal{Z}_{g,1}$. \square

Relation to work of J. Murakami and O. Viro. The representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ at $q^4 = 1$ determines a degenerate instance of the Chern-Simons topological field theory that has been related to the Alexander polynomial [31, 46]. The Jones-Wenzl projector $p_3 \in \text{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes 3})$ takes the form:

$$p_3 = \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \frac{d}{d^2 - 1} \left(\left| \begin{array}{c} \diagup \\ | \\ \diagdown \end{array} \right| + \left| \begin{array}{c} \diagdown \\ | \\ \diagup \end{array} \right| \right) + \frac{1}{d^2 - 1} \left(\left| \begin{array}{c} \diagup \\ \diagdown \end{array} \right| + \left| \begin{array}{c} \diagdown \\ \diagup \end{array} \right| \right).$$

where $d = q + q^{-1}$. Taking $q = \sqrt{-1}$, gives $d = 0$ and $d^2 - 1 = -1$. This eliminates the middle term above, leaving the bypass triangle

$$p_3 = \left| \begin{array}{c} | \\ | \\ | \end{array} \right| - \left| \begin{array}{c} \diagup \\ \diagdown \end{array} \right| - \left| \begin{array}{c} \diagdown \\ \diagup \end{array} \right|.$$

Since the righthand side is zero in Theorem 5.16, there is only a relationship between the contact geometry and representation theory after reducing by the Goodman-Wenzl ideal $\langle p_3 \rangle$ [10, Appendix].

6. COMPARISON BETWEEN CATEGORIES ASSOCIATED TO DISKS

In this section we show that the categories associated to the disk $(D^2, 2n)$ with $2n$ points by the Heegaard-Floer theory $\mathcal{A}(D^2, 2n)$, the contact topology $\mathcal{Co}(D^2, 2n)$ and the formal contact construction are Morita equivalent.

$$\mathcal{A}(D^2, 2n) \cong \mathcal{Co}(D^2, 2n) \cong \mathcal{Ko}_+(D^2, 2n)$$

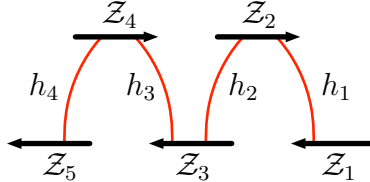
This is accomplished by choosing an arc parameterization \mathcal{M}_n of the disk $(D^2, 2n)$ so that the associated Heegaard-Floer category $\mathcal{A}(D^2, 2n) \cong \mathcal{A}(-\mathcal{M}_n)$ has the same quiver presentation as the algebraic contact category $\mathcal{Co}(D^2, 2n) \cong \mathcal{Y}_n$ studied by Y. Tian. This equivalence is combined with Theorem 5.13 to show that both categories are Morita equivalent to the positive half of the formal contact category $\mathcal{Ko}_+(D^2, 2n)$.

6.1. The Heegaard-Floer categories associated to a disk. In this section an arc diagram \mathcal{M}_n and an arc parameterization of the disk $(D^2, 2n)$ with $2n$ marked points by \mathcal{M}_n are introduced. The Bordered Sutured Floer theory developed by R. Zarev associates a dg category $\mathcal{A}(\mathcal{M}_n)$ to this parameterization. In Section 6.14, we will find that this category is the same as Y. Tian's quiver algebra \mathcal{R}_n . The disk will be oriented in the opposite direction of later sections.

Definition 6.2. The *zig-zag arc diagram* \mathcal{M}_n is defined inductively as follows:

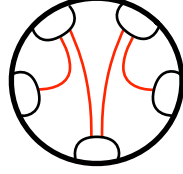
- (1) The arc diagram \mathcal{M}_2 consists of two lines $Z = \{Z_1, Z_2\}$ and two points $\mathbf{a} = \{a_1, a'_1\}$ where, $a_1 \in Z_1$, $a'_1 \in Z_2$ and $M(a_1) = M(a'_1)$.
- (2) If n is odd then \mathcal{M}_n is obtained from \mathcal{M}_{n-1} by adding a new line Z_n , containing the point a_{n-1} , to the right of the line Z_{n-2} and adding the point a'_{n-1} to the line Z_{n-1} immediately to the left of a'_{n-2} .
- (3) If n is even then \mathcal{M}_n is obtained from \mathcal{M}_{n-1} by adding a new line Z_n , containing the point a'_{n-1} , to the left of Z_{n-2} and then adding the point a_{n-1} to Z_{n-1} to the right of the point a_{n-2} .

If we imagine the line segments $\{Z_i\}_{i=1}^n$ to be embedded sequentially along the real line \mathbb{R} then an orientation on each line segment is induced by choosing an orientation of \mathbb{R} ; they all point either to the left or to the right. The name zig-zag becomes clear after rearranging the line segments into a zig-zag pattern.



The arc diagram for \mathcal{M}_5 is pictured above. The line labelled Z_i is the i th line segment in the construction from Definition 6.2. The lines h_i connect the matched pairs $\{a_i, a'_i\}$. If the illustration above is understood to specify an embedding of the

arc diagram into the plane then thickening each of the components produces the parameterization of the disk $(D^2, 2 \cdot 5)$ with 10 points pictured below.



Giving the plane the standard $\langle x, y \rangle$ orientation induces an orientation on $(D^2, 2n)$ in which the boundary is oriented clockwise.

Proposition 6.3. *The dg category $\mathcal{A}(-\mathcal{M}_n)$ has trivial differential: $d = 0$.*

Proof. By construction, as an algebra with idempotents, the dg category $\mathcal{A}(\mathcal{M}_n)$ is a subalgebra of a tensor product of copies of strands algebras $\mathcal{A}(1)$ and $\mathcal{A}(2)$. Neither of these algebras have differentials. Any tensor product of algebras without differentials does not have a differential. Any subalgebra of an algebra without differential does not have a differential, see also [50, Prop. 9.2]. \square

Without a differential, the category $\mathcal{A}(-\mathcal{M}_n)$ can be given a presentation by following the discussion in [50, §2.3]. The remainder of this section contains of a derivation of these generators and relations.

First note that the idempotents in this construction correspond to the objects of the category $\mathcal{A}(-\mathcal{M}_n)$, the idempotents are indexed by a choice of a subset

$$S \subset \{h_1, \dots, h_{n-1}\}$$

of the 1-handles which identify matched pairs in the arc diagram \mathcal{M}_n [50, Def. 2.5].

Of the line segments $\{\mathcal{Z}_1, \dots, \mathcal{Z}_n\}$, there are $n - 2$ line segments $\{\mathcal{Z}_2, \dots, \mathcal{Z}_{n-1}\}$ containing two points of the form:

$$a'_k a'_{k-1} \text{ for } 1 \leq k \leq n - 2 \quad \text{or} \quad a_k a_{k+1} \text{ for } 1 < k \leq n - 1 \quad (6.1)$$

when n is even. When n is odd:

$$a'_k a'_{k-1} \text{ for } 1 \leq k \leq n - 1 \quad \text{or} \quad a_k a_{k+1} \text{ for } 1 < k \leq n - 2. \quad (6.2)$$

There are only non-trivial elements in the parts of $\mathcal{A}(-\mathcal{M}_n) \subset \mathcal{A}(1) \otimes \mathcal{A}(2)^{\otimes n-2} \otimes \mathcal{A}(1)$ corresponding to the $\mathcal{A}(2)$ -tensor factors. Each such factor contains a Reeb chord $\rho_{k,k+1}$ or $\rho_{k+1,k}$. If the line segment contains the points $a'_{k+1} a'_k$ then the Reeb chord $\rho_{k,k+1}$ connects $\rho_{k,k+1}^- = a'_k$ to $\rho_{k+1,k}^+ = a'_{k+1}$. If the line segment contains the points $a_k a_{k+1}$ then the Reeb chord $\rho_{k+1,k}$ connects $\rho_{k+1,k}^- = a_{k+1}$ to $\rho_{k,k+1}^+ = a_k$. Since the k th 1-handle h_k corresponds to the matching of the pair a_k and a'_k , the Reeb chords $\rho_{k,k+1}$ and $\rho_{k+1,k}$ correspond to maps:

$$\rho_{k,k+1} : h_k \rightarrow h_{k+1} \quad \text{and} \quad \rho_{k+1,k} : h_{k+1} \rightarrow h_k \quad (6.3)$$

Translating Equations (6.1) and (6.2) above into the language of Equation (6.3) tells us when such maps can be found in the category $\mathcal{A}(-\mathcal{M}_n)$. If n is even then there are maps:

$$h_{n-1} \xrightarrow{\rho_{n-1,n-2}} h_{n-2} \xleftarrow{\rho_{n-3,n-2}} h_{n-3} \rightarrow \cdots \leftarrow h_3 \xrightarrow{\rho_{3,2}} h_2 \xleftarrow{\rho_{1,2}} h_1$$

and if n is odd then there are maps:

$$h_{n-1} \xleftarrow{\rho_{n-2,n-1}} h_{n-2} \xrightarrow{\rho_{n-2,n-3}} h_{n-3} \leftarrow \cdots \leftarrow h_3 \xrightarrow{\rho_{3,2}} h_2 \xleftarrow{\rho_{1,2}} h_1.$$

Increasing the number n by one has the effect of adding one new Reeb chord.

The generators of the full category $\mathcal{A}(-\mathcal{M}_n)$ are obtained by extending each Reeb chord by identity in all possible ways [50, Def. 2.9]. In more detail, if $S = h_{i_1} h_{i_2} \cdots h_{i_j} \cdots h_{i_{k-1}} h_{i_k}$ is a subset of 1-handles which have been ordered so that $i_j < i_{j+1}$ then there is a generator:

$$h_{i_1} h_{i_2} \cdots h_{i_j} \cdots h_{i_{k-1}} h_{i_k} \rightarrow h_{i_1} h_{i_2} \cdots h_{i_j \pm 1} \cdots h_{i_{k-1}} h_{i_k} \quad (6.4)$$

in $\mathcal{A}(-\mathcal{M}_n)$ when there is a Reeb chord $\rho_{i_j, i_j \pm 1} : h_{i_j} \rightarrow h_{i_j \pm 1}$ as above and the 1-handle $h_{i_j \pm 1}$ isn't contained in set S :

$$i_{j \pm 1} \notin \{i_1, i_2, \dots, i_k\}.$$

None of the relations satisfied by the strands algebras apply in our context because the Reeb chords are contained in independent strands algebras $\mathcal{A}(2)$ of order two. There are only is only one relevant family of relations, stemming from the observation that maps applied to independent tensor factors commute.

$$\begin{array}{ccc} & \cdots h_{i_j \pm 1} \cdots h_{i_\ell} \cdots & \\ \nearrow & & \searrow \\ \cdots h_{i_j} \cdots h_{i_\ell} \cdots & & \cdots h_{i_j \pm 1} \cdots h_{i_\ell \pm 1} \cdots \\ \searrow & & \nearrow \\ & \cdots h_{i_j} \cdots h_{i_\ell \pm 1} \cdots & \end{array} \quad (6.5)$$

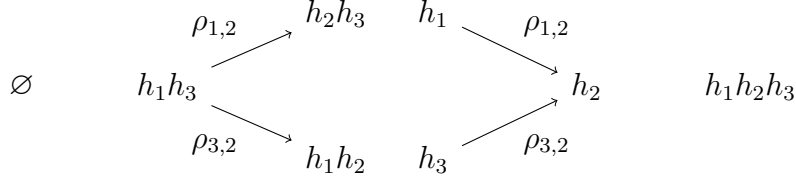
Said differently, whenever generators can be applied out-of-order to form a square, as pictured above, this square must commute.

The examples below will be compared to Examples 6.10 and 6.11 in Section 6.14 later.

Example 6.4. The structure of $\mathcal{A}(-\mathcal{M}_3)$ can be pictured in the following way:

$$\emptyset \quad h_1 \xrightarrow{\rho_{1,2}} h_2 \quad h_1 h_2$$

Example 6.5. The structure of $\mathcal{A}(-\mathcal{M}_4)$ is illustrated by the diagram below:



Remark 6.6. It follows invariance of the general Bordered Sutured theory that the dg categories associated to a surface by any two parameterizations are Morita equivalent [51]. In particular, there is an arc parameterization \mathcal{W}_n [50, Ex. 9.1] for which there is an isomorphism of dg categories $\mathcal{A}(\mathcal{W}_n) \cong \mathcal{A}(n-1)^{\text{op}}$ [50, Prop. 9.1], where $\mathcal{A}(n-1)$ is the strands algebra [26, §3.1]. Therefore, $\mathcal{A}(-\mathcal{M}_n) \cong \mathcal{A}(n-1)^{\text{op}}$ in Hmo.

6.7. The contact category associated to a disk. Here we introduce the category \mathcal{Y}_n that Y. Tian associates to the disk with $2n$ boundary points [41]. We will not discuss gradings.

Indexing multicurves with nil-Temperley-Lieb notation. Monomials in the nil-Temperley-Lieb algebra, will be used to denote multicurves $\gamma \subset (D^2, 2n)$ in the disk. In particular, multicurves determined by monomials $e_{i_1}e_{i_2} \cdots e_{i_k}$, which have been ordered, so as to satisfy $i_1 < i_2 < \cdots < i_k$, correspond to the objects in Y. Tian's construction, see Definition 6.9.

Definition 6.8. The *nil-Temperley-Lieb algebra* \mathcal{N}_n is the k -algebra on generators: e_i , $1 \leq i < n$, subject to the relations:

- (1) $e_i^2 = 0$ for $1 \leq i < n$
- (2) $e_i e_j = e_j e_i$ for $|i - j| > 2$ and
- (3) $e_i e_{i \pm 1} e_i = e_i$.

If the ground ring k is changed to $\mathbb{Z}[q, q^{-1}]$ and the first relation is changed from $e_i^2 = 0$ to $e_i^2 = q + q^{-1}$ then the algebra \mathcal{N}_n introduced above becomes the well-known Temperley-Lieb algebra \mathcal{TL}_n , see [20].

The relationship between the Temperley-Lieb algebra and the planar algebra of multicurves extends to the nil-variant \mathcal{N}_n introduced above. There is a basis for the algebra \mathcal{N}_n consisting of monomials which is in one-to-one correspondence with isotopy classes of boundary connected multicurves in a pointed oriented disk $(D^2, 2n)$. This can be seen after each generator e_i is identified with a multicurve $\gamma(e_i)$.

$$e_i \mapsto \gamma(e_i)$$

If the disk is pictured so that the first n points are situated on the top of the disk and the last n points are situated on the bottom of the disk then all of the strands of

$\gamma(e_i)$ are vertical except for two which connect the i th and $(i + 1)$ -st points in each collection. The products, $\gamma(e_i e_j) = \gamma(e_i) \gamma(e_j)$, of generators correspond to vertically stacking the multicurves. For instance, when $n = 3$ we have the following pictures:

$$\gamma(1) = \left| \begin{array}{c} | \\ | \\ | \end{array} \right|, \quad \gamma(e_1) = \begin{array}{c} \cup \\ \cap \end{array} \left| \begin{array}{c} | \\ | \\ | \end{array} \right| \quad \text{or} \quad \gamma(e_1 e_2) = \begin{array}{c} \cup \quad \cup \\ \cap \quad \cap \end{array}.$$

In the image of the map γ , the second and third relations in Definition 6.8 correspond to isotopy and the first relation implies that any multicurve containing a homotopically trivial component is zero.

This observation can be used to construct a set map γ from the monomials the nil-Temperley-Lieb algebra \mathcal{N}_n to positive dividing sets on $(D^2, 2n)$. Since all of the defining relations for \mathcal{N}_n preserve monomiality: the product of monomials is a monomial and each monomial $x \in \mathcal{N}_n$ corresponds to a multicurve $\gamma(x)$. After signing the regions of $D^2 \setminus \gamma(x)$, this determines a dividing set on the disk. Knowledge of the map γ is assumed throughout the next section.

Y. Tian's disk category. Y. Tian's category \mathcal{Y}_n is introduced by the sequence of definitions below. The construction presented here is equivalent to the original [41]. However, we will use the algebra \mathcal{N}_n to express the presentation in a more familiar notation.

Definition 6.9. There is a quiver \mathcal{Q}_n with vertices given by ordered monomials:

$$e_S = e_{i_1} e_{i_2} \cdots e_{i_k} \in \mathcal{N}_n \quad \text{where} \quad S = \{i_1 < i_2 < \cdots < i_k\}.$$

and $1 \leq i_j < n$ for $j = 1, \dots, k$. There is a directed edge

$$\theta_p : e_S \rightarrow e_T$$

from e_S to e_T when the set T can be obtained from the set S by adjoining the disjoint subset $\{p, p + 1\}$. (In symbols: $|T| = |S| + 2$ and $T = S \cup \{p, p + 1\}$.)

Before introducing the category \mathcal{Y}_n , the definition above is illustrated by the examples below.

Example 6.10. When $n = 3$, the quiver \mathcal{Q}_3 assumes a rather unassuming form:

$$e_1 \quad 1 \xrightarrow{\theta_1} e_1 e_2 \quad e_2$$

Example 6.11. When $n = 4$, the quiver \mathcal{Q}_4 is more complicated:

$$\begin{array}{ccccccc} & & & e_1 e_2 & e_1 & \theta_2 & \\ & & \theta_1 \nearrow & & & \searrow & \\ e_1 e_3 & 1 & & & & & e_1 e_2 e_3 & e_2 \\ & & \theta_2 \searrow & & & \nearrow & \\ & & e_2 e_3 & e_3 & \theta_1 & & \end{array}$$

Each arrow $\theta_p : e_S \rightarrow e_T$ corresponds to a bypass move $\gamma(e_S) \rightarrow \gamma(e_T)$ between the multicurves $\gamma(e_S)$ and $\gamma(e_T)$, involving the p th and $p + 1$ st regions in the disk, see Equation (6.6).

The disk category \mathcal{R}_n is the category generated by the graph \mathcal{Q}_n , modulo the relation that compositions of disjoint bypass moves commute.

Definition 6.12. The *disk category* \mathcal{R}_n is the k -linear category generated by the graph \mathcal{Q}_n subject to the relations:

$$\theta_p \theta_q = \theta_q \theta_p \quad \text{for each pair of arrows} \quad \theta_p \theta_q, \theta_q \theta_p : e_S \rightarrow e_T \text{ in } \mathcal{Q}_n.$$

The disk category \mathcal{R}_n can be viewed as a dg category with $d = 0$. Recall the notion of pretriangulated hull from Section 2.1.

Definition 6.13. The category \mathcal{Y}_n associated to the disk $(D^2, 2n)$ is the pretriangulated hull of the disk category \mathcal{R}_n :

$$\mathcal{Y}_n = \mathcal{R}_n^{\text{pretr}}.$$

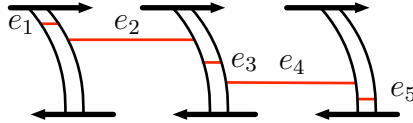
6.14. Relationship between the contact category and the Heegaard-Floer category. Here we show that the category $\mathcal{A}(-\mathcal{M}_n)$ found in Section 6.1 is isomorphic to Y. Tian's disk category \mathcal{R}_n from Section 6.7.

Theorem 6.15.

$$\mathcal{R}_n \xrightarrow{\sim} \mathcal{A}(-\mathcal{M}_n)$$

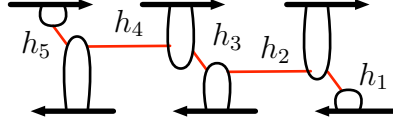
Proof. The similarities between Examples 6.4, 6.10 and Examples 6.5, 6.11 are suggestive. We will discuss the case when n is even, the case when n is odd is similar. We first give a bijective correspondence between the objects in either category. After this the generators in either category are related to one another by representing each by a geometric bypass moves.

There is a one-to-one correspondence between the objects in each category. In this correspondence the identity diagram $1 \in \mathcal{N}_n$ corresponds to selecting all of the odd 1-handles: $h_1 h_3 \cdots h_{n-1}$. Suppose that $w = e_{i_1} e_{i_2} \cdots e_{i_k} \in \mathcal{N}_n$ is an ordered monomial. Then to construct the selection of 1-handles associated to w we cut the identity diagram along the odd generators e_{2p+1} in w and glue 1-handles along the even generators e_{2q} as indicated by the picture below:



There is a uniquely determined set $S \subset \{h_1, \dots, h_{n-1}\}$ of 1-handles which remain after performing this operation and so there is a map from ordered monomials to subsets S of the set of 1-handles.

Conversely, the empty set of 1-handles \emptyset corresponds to the product of the even generators $e_2 e_4 \dots e_n \in \mathcal{N}_n$. If $h_{i_1} h_{i_2} \dots h_{i_k}$ is an arbitrary selection of 1-handles then gluing each 1-handle h_{i_j} into the picture below, in the indicated fashion, uniquely determines a multicurve associated to a positive monomial.



The maps introduced above are inverse. There is a bijection between the objects of either category.

If $w, w' \in \mathcal{N}_n$ are ordered monomials then an arrow $\theta_p : ww' \rightarrow we_p e_{p+1} w'$ in the graph \mathcal{Q}_n corresponds to the bypass move $\theta_p : \gamma(ww') \rightarrow \gamma(we_p e_{p+1} w')$ pictured below,

$$\theta_p = \begin{array}{c} \text{Diagram of a bypass move: two vertical lines with horizontal segments at the top and bottom, connected by a horizontal dashed line in the middle.} \end{array} \quad (6.6)$$

For example, after a rotation, the only arrow in the quiver \mathcal{Q}_3 corresponds to the bypass illustrated before Definition 3.12. On the other hand, the basic Reeb chords: $\rho_{k,k+1} : h_k \rightarrow h_{k+1}$ and $\rho_{k+2,k+1} : h_{k+2} \rightarrow h_{k+1}$ from Section 6.1 correspond to the pictures:

$$\begin{array}{c} \text{Diagram 1: A horizontal line with a semi-circle above it, connected by a dashed line to another horizontal line with a semi-circle above it.} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 2: A horizontal line with a semi-circle above it, connected by a dashed line to another horizontal line with a semi-circle above it.} \end{array} \quad (6.7)$$

so that the two combinatorial notions perform the same function between multicurves in the correspondence between the objects.

There are no relations in either category besides the commutativity of diagrams in Equation 6.5 and Definition (6.12). \square

6.16. Relationship between the disk category and the formal contact category. In this section we will construct a Morita equivalence between the Heegaard-Floer category $\mathcal{A}(-\mathcal{M}_n)$ considered in Section 6.1 and the formal contact category $\mathcal{K}o_+(D^2, 2n)$.

The discussion in prior sections suffices to define a functor:

$$\mu : \mathcal{A}(-\mathcal{M}_n) \rightarrow \mathcal{K}o_+(D^2, 2n).$$

To each collection of 1-handles $C = h_{i_1} h_{i_2} \cdots h_{i_k}$ we associate the elementary generator $\mathfrak{z}_C \in \text{Ob}(\text{Pre-}\mathcal{K}o_+(D^2, 2n))$. The basic Reeb chords correspond to the bypass moves pictured in Equation (6.7) above. Composing this functor with the quotient map $Q : \text{Pre-}\mathcal{K}o_+(D^2, 2n) \rightarrow \mathcal{K}o_+(D^2, 2n)$ yields μ above.

Theorem 6.17. *The functor $\mu : \mathcal{A}(-\mathcal{M}_n) \rightarrow \mathcal{K}o_+(D^2, 2n)$ determines a Morita equivalence.*

The proof of the theorem will use the fact that if \mathcal{A} and \mathcal{C} are small dg categories then \mathcal{A} is Morita equivalent to \mathcal{C} when \mathcal{C} is quasi-equivalent to a full dg subcategory \mathcal{B} of the category of \mathcal{A} whose objects form a set of small generators. This is a special case of a more general statement [21, Thm. 8.2].

Proof. Using Theorem 5.13, it suffices to check that for each pair of collections of 1-handles C, C' the maps:

$$\mu_{C, C'} : \text{Hom}_{\mathcal{A}(-\mathcal{M}_n)}(C, C') \rightarrow \text{Hom}_{\mathcal{K}o_+(D^2, 2n)}(\mathfrak{z}_C, \mathfrak{z}_{C'})$$

are quasi-isomorphisms. Since the trivial bypasses must bound caps and are removed by relation (1) in Definition 3.16. The only bypasses $\mathfrak{z}_C \rightarrow \mathfrak{z}_{C'}$ between elementary generators are those that appear in Equation (6.7). These bypasses and their compositions are the cycles in $\text{Pre-}\mathcal{K}o_+(D^2, 2n)$. It suffices to show that they remain cycles in the quotient.

The remainder follows from the commutativity of pushouts:

$$L_S L_{S'} \mathcal{C} \cong L_{S \sqcup S'} \mathcal{C} \cong L_{S'} L_S \mathcal{C}$$

and the observation that the maps $Q_{C, C'} : \text{Hom}_{\mathcal{C}}(C, C') \rightarrow \text{Hom}_{L_S \mathcal{C}}(C, C')$ are quasi-isomorphisms for any single Postnikov localization. The latter can be seen by identifying a single Postnikov localization as an instance of Drinfeld localization under the Yoneda embedding, see Proposition 2.23. The Drinfeld localization modifies the homological structure of the morphisms by adding a single map h which is a boundary $dh = 1_K$ where K is as in the proof of Proposition 2.23. This makes any cycle to or from K into a boundary, but does not create any other boundaries. Since K is not an elementary generator \mathfrak{z}_C for some C , the result follows. \square

6.18. Dualities. Our discussion concludes with some mention of dualities. In Examples 6.10 and 6.11, duality is found in the lateral symmetry of the graph \mathcal{Q}_n . If $[n]$ denotes an ordered set $\{1 < 2 < \cdots < n\}$ then the assignment:

$$e_S^y = e_{[n] \setminus S}$$

determines a contravariant involution:

$$-^y : \mathcal{Y}_n^{\text{op}} \rightarrow \mathcal{Y}_n.$$

In \mathcal{Y}_n there are no signed regions and the lateral symmetry is contravariant; so the functor $-^y$ cannot directly correspond to a functor, such as $-^\vee$, between formal contact categories. The proposition records the correct formulation. The proof is left to the reader.

Proposition 6.19. *The diagram below commutes,*

$$\begin{array}{ccc} \mathcal{Y}_n^{\text{op}} & \xrightarrow{\mathbf{i}_+^{\text{op}}} & Ko_+(D^2, 2n)^{\text{op}} \\ \downarrow -^y & & \downarrow \alpha \\ \mathcal{Y}_n & \xrightarrow{\mathbf{i}_+} & Ko_+(D^2, 2n) \end{array}$$

where the functor $\alpha = (-)^\vee \circ (-) \circ (r^{-1})^{\text{op}}$ is the composition of three equivalences: r is the element of the mapping class group which rotates the basepoint z by one region clockwise (Corollary 5.4), $(-)$ reverses the orientation of the disk (Proposition 4.8) and $(-)^\vee$ changes the signs of the regions (Proposition 4.7).

7. LINEAR BORDERED HEEGAARD FLOER CATEGORIES

Within the framework of the Bordered Heegaard Floer theory, a differential graded category $\mathcal{A}(\mathcal{Z})$ is associated to each arc diagram \mathcal{Z} . For some choices this category satisfies $d = 0$ and it is possible to write down a quiver presentation. In this section, these categories are related to the corresponding formal contact categories. Functors are defined:

$$\begin{aligned} \mathcal{A}(-\mathcal{Z}_{0,n}, 2-n) &\xrightarrow{\sigma_n} Ho(\mathcal{K}o_+^{2n-4}(\Sigma_{0,n}, n \cdot 2)) \quad \text{and} \\ \mathcal{A}(-\mathcal{Z}_{g,1}, 1-g) &\xrightarrow{\tau_g} Ho(\mathcal{K}o_+^{2g-2}(\Sigma_{g,1}, 2)) \end{aligned}$$

where $\mathcal{Z}_{0,n}$ and $\mathcal{Z}_{g,1}$ are arc diagrams which parameterize surfaces, $\Sigma_{0,n}$ and $\Sigma_{g,1}$, of genus zero with n boundary components and of genus g with one boundary component respectively. We fix two points on every boundary component and require that $n > 1$ and $g > 0$. For details concerning these algebras, see [50, §2], [28, §2] or [26, §3].

7.1. A surface $\Sigma_{0,n}$ of genus 0 with several boundary components. When n disks are removed from the 2-sphere

$$\Sigma_{0,n} = S^2 \setminus \bigcup_{k=1}^n D^2 \quad n > 1,$$

and two points are fixed on each of its boundary components, the resulting surface can be parameaterized by the arc diagram $\mathcal{Z}_{0,n}$ found in the definition below.

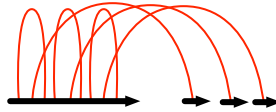
Definition 7.2. The arc diagram $\mathcal{Z}_{0,n}$ consists of n oriented line segments $Z = \{Z_1, Z_2, \dots, Z_n\}$. On the first line segment Z_1 there are $3n - 3$ points and there is one point on each of the remaining line segments $\{Z_2 \dots Z_n\}$:

$$Z_1 = a_1 b_1 a'_1 a_2 b_2 a'_2 \cdots a_{n-1} b_{n-1} a'_{n-1} \quad \text{and} \quad Z_k = b'_k \text{ for } 2 \leq k \leq n.$$

The set of points is given by $\mathbf{a} = \{a_k, a'_k, b_k, b'_k : 1 \leq k < n\}$. The line Z_1 is oriented so that the subscripts of the points increase in value. The matching function is determined by the rules: $M(a_k) = M(a'_k)$ and $M(b_k) = M(b'_k)$.

The annulus $\Sigma_{0,2}$ and its parameterization by $\mathcal{Z}_{0,2}$ are pictured in Example 5.10.

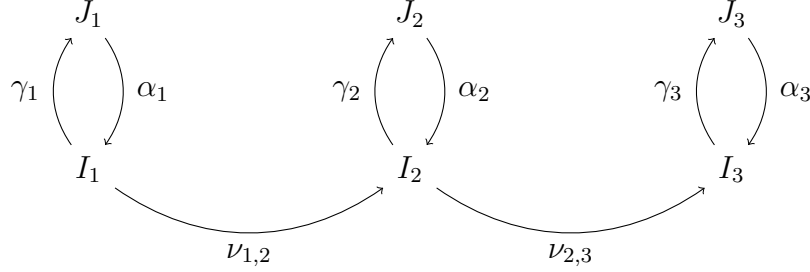
Example 7.3. When $n = 4$, the definition above is illustrated by the picture below:



Definition 7.4. The category $\mathcal{A}(-\mathcal{Z}_{0,n}, 2-n)$ associated to the arc diagram $\mathcal{Z}_{0,n}$ is the k -linear category determined by a quiver with vertices: I_k and J_k corresponding to the pairs $\{a_k, a'_k\}$ and $\{b_k, b'_k\}$ for $1 \leq k < n$ respectively. There are arrows $\alpha_k : I_k \rightarrow J_k$, $\gamma_k : J_k \rightarrow I_k$ and $\nu_{k,k+1} : I_k \rightarrow I_{k+1}$ subject to the relations:

- (1) $\alpha_k \gamma_k = 0 : J_k \rightarrow J_k$ and
- (2) $\nu_{k+1,k+2} \nu_{k,k+1} = 0 : I_k \rightarrow I_{k+2}$.

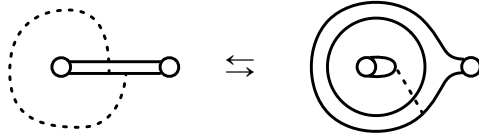
Example 7.5. The quiver underlying the category $\mathcal{A}(-\mathcal{Z}_{0,4}, -2)$ in the definition above is illustrated below.



The construction of the functor $\sigma_n : \mathcal{A}(\mathcal{Z}_{0,n}, 2-n) \rightarrow Ho(\mathcal{K}o_+^{2n-4}(\Sigma_{0,n}, n \cdot 2))$ will occur in two stages.

First note that the parameterization of $\Sigma_{0,n}$ by the arc diagram allows us to associate to each object, I_k or J_k , $1 \leq k < n$, a dividing set contained in an annulus. In fact, Theorem 5.13 states that these dividing sets generate the contact category. In each annulus we will describe bypass moves corresponding to the arrows $\alpha_k : I_k \rightarrow J_k$ and $\gamma_k : J_k \rightarrow I_k$. We will check that these bypass moves satisfy the first collection of relations in the definition above. After this has been done, bypass moves corresponding to the lateral arrows $\nu_{k,k+1} : I_k \rightarrow I_{k+1}$ will be introduced and shown to satisfy the second collection of relations.

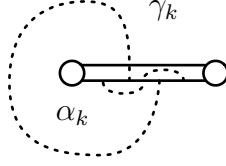
Step #1. For each annulus, the dividing sets J_k , I_k , and the bypass moves corresponding to the maps $\gamma_k : J_k \rightarrow I_k$ and $\alpha_k : I_k \rightarrow J_k$ can be depicted by the curves:



The dividing set associated to J_k is featured on the lefthand side and the dividing set associated to I_k is shown on the righthand side. The map γ_k runs from left to right and the map α_k runs from right to left. The equators of γ_k and α_k are determined by the dashed lines in the dividing sets corresponding to J_k and I_k respectively.

Proposition 7.6. *The relation $\alpha_k \gamma_k = 0$ holds in the formal contact category $Ho(\mathcal{K}o_+(S^1 \times [0, 1], (2, 2)))$.*

Proof. The map $\alpha_k \gamma_k : J_k \rightarrow J_k$ is a composition of two disjoint bypass moves. This is illustrated below.



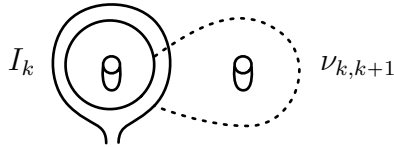
Relation (2) in the Definition 3.18 of the formal contact category implies that applying the two bypass moves in either order must commute:

$$\begin{array}{ccc}
 J_k & \xrightarrow{\gamma_k} & I_k \\
 \downarrow & & \downarrow \alpha_k \\
 0 & \longrightarrow & J_k,
 \end{array}$$

but performing the bypass move α_k before the bypass move γ_k must be zero since α_k is capped. \square

The same argument shows that one of the terms in the commutative diagram associated to the other composition $\gamma_k \alpha_k$ is a capped bypass equivalent to identity.

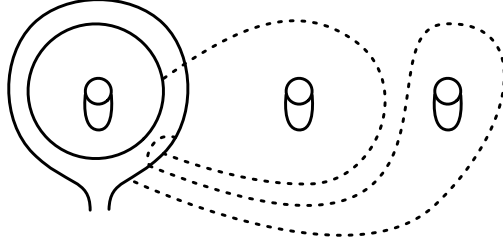
Step #2. As pictured above, the idempotents I_k correspond to the boundaries of regular neighborhoods of loops about each boundary component of $\Sigma_{0,n}$. We think of $\Sigma_{0,n}$ as a subset of the plane $D^2 \setminus \sqcup_{k=1}^{n-1} D^2 \subset \mathbb{R}^2$ with $n - 1$ disks removed from its interior. The arc parameterization orders the boundary components and the associated idempotents. When two of them are adjacent, I_k and I_{k+1} , there is a bypass move $\nu_{k,k+1} : I_k \rightarrow I_{k+1}$ determined by the equator of the bypass disk in the illustration below.



Proposition 7.7. *The relation: $\nu_{k+1,k+2} \nu_{k,k+1} = 0$ holds in the formal contact category $Ho(Ko_+(\Sigma_{0,n}, n \cdot 2))$.*

Proof. The proof is analogous to the proof of Proposition 7.6 above. The bypass moves representing $\nu_{k,k+1}$ and $\nu_{k+1,k+2}$ are disjoint. Considering them simultaneously

produces visual aid below.



The curve on the far right represents the equator of the bypass $\nu_{k+1,k+2}$. Since this bypass move is capped the composition factors through zero. \square

Y. Tian's annulus. As in Section 6 above, in Y. Tian's work [42, §2.2], the category associated to an annulus with two points on each boundary component is the pretriangulated hull on the free k -linear category associated to a quiver with five vertices: I , E , F and EF . The dividing sets associated to E and F are Euler dual and are neither the source nor the target of any non-trivial edges. There are two dividing sets I and EF generating the subcategory with Euler number zero via maps $\gamma : I \rightarrow EF$ and $\alpha : EF \rightarrow I$ which are required to satisfy the relation:

$$\alpha\gamma = 0.$$

This description is summarized by the illustration below.

$$F \qquad \qquad \gamma : I \rightleftarrows EF : \alpha \qquad \qquad E$$

The quiver in the center is precisely $\mathcal{A}(-\mathcal{Z}_{0,2}, 0)$ above.

Remark 7.8. It is natural to ask about surfaces $\Sigma_{0,n}$ with $n > 2$. There are presently two constructions in the literature. In [42], the category associated to $\Sigma_{0,n}$ is a Bordered Heegaard Floer category by definition. Precisely the same can be said for the categories considered by I. Petkova and V. Vértesi [35]. While the former chooses an arc parameterization which yields a heart encoding contact geometry, the latter chooses an arc parameterization which yields a Stendhalic extension [48] of the strands algebra [26]. In both cases the arc parameterizations are *degenerate* so that the Border Heegaard Floer construction does not suffice to imply an equivalence between the two and the materials here do not necessarily apply.

7.9. A surface $\Sigma_{g,1}$ of genus g with one boundary component.

Definition 7.10. The arc diagram $\mathcal{Z}_{g,1}$ consists of $4g$ points $\mathbf{a} = \{a_k, a'_k, b_k, b'_k : 1 \leq k \leq g\}$ on one line segment $Z = \{\mathcal{Z}_1\}$

$$\mathcal{Z}_1 = a_1 b_1 a'_1 b'_1 a_2 b_2 a'_2 b'_2 \cdots a_g b_g a'_g b'_g$$

which is oriented so that the indices above are increasing. The matching function is determined by the rules $M(a_k) = M(a'_k)$ and $M(b_k) = M(b'_k)$ for $1 \leq k \leq g$.

Example 7.11. The arc diagram $\mathcal{Z}_{2,1}$ is illustrated below.



Definition 7.12. The category $\mathcal{A}(-\mathcal{Z}_{g,1}, 1-g)$ associated to the arc diagram $\mathcal{Z}_{g,1}$ is the k -linear category determined by a quiver with vertices: I_k and J_k corresponding to the pairs $\{a_k, a'_k\}$ and $\{b_k, b'_k\}$ for $1 \leq k \leq g$ respectively. There are arrows:

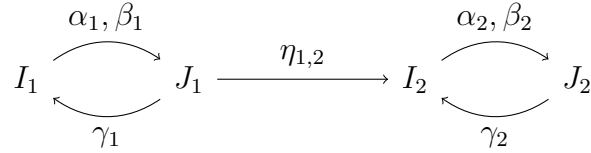
$$\alpha_k, \beta_k : I_k \rightarrow J_k, \gamma_k : J_k \rightarrow I_k \quad \text{and} \quad \eta_{k,k+1} : I_k \rightarrow J_{k+1},$$

the compositions of which satisfy the relations below:

- (1) $\alpha_k \gamma_k = 0 : J_k \rightarrow J_k$ and $\gamma_k \beta_k = 0 : I_k \rightarrow I_k$
- (2) $\eta_{k,k+1} \alpha_k = 0 : I_k \rightarrow I_{k+1}$ and $\beta_k \eta_{k,k+1} = 0 : J_k \rightarrow J_{k+1}$

Note that $\eta_{k,k+1} : I_k \rightarrow J_{k+1}$ is not the same as $\nu_{k,k+1} : I_k \rightarrow I_{k+1}$ in the previous section.

Example 7.13. The quiver underlying the construction of the category $\mathcal{A}(-\mathcal{Z}_{2,1}, -1)$ is illustrated below.

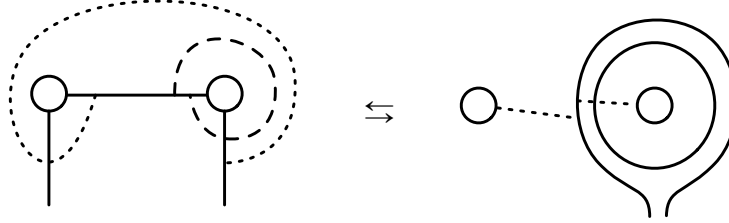


The construction of the functor $\tau_g : \mathcal{A}(-\mathcal{Z}_{g,1}, 1-g) \rightarrow \mathcal{K}o^{2g-2}(\Sigma_{g,1}, 2)$ will occur in two stages.

First note that the parameterization of $\Sigma_{g,1}$ by the arc diagram allows us to associate to each k , $1 \leq k \leq g$, a pair of dividing sets I_k and J_k contained in a torus $\Sigma_{1,1} \subset \Sigma_{g,1}$ with one boundary component. In fact, Theorem 5.13 states that these dividing sets generate the category. In each torus, we will describe bypass moves corresponding to the arrows $\alpha_k, \beta_k : I_k \rightarrow J_k$ and $\gamma_k : J_k \rightarrow I_k$ and check that these bypass moves satisfy the first collection of relations in the definition above.

After this has been done, bypass moves corresponding to the lateral arrows $\eta_{k,k+1}$ will be introduced and shown to satisfy the second collection of relations.

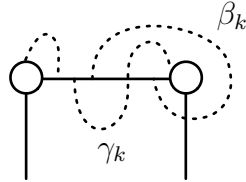
Step #1. For each torus, the dividing sets I_k , J_k , and the bypass moves corresponding to the maps $\alpha_k, \beta_k : I_k \rightarrow J_k$ and $\gamma_k : J_k \rightarrow I_k$ can be depicted by the curves:



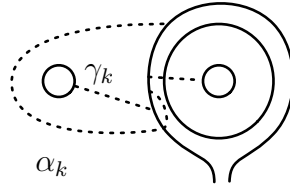
On either side of the arrows in the picture above, the two small circles are identified by folding the page to form the surface $(T^2 \setminus D^2, 2)$. The dividing set associated to I_k is featured on the lefthand side and the dividing set associated to J_k is featured on the righthand side. The maps α_k and β_k run from left to right and the map γ_k runs from right to left. The equator of the map α_k is dotted and the equator of β_k is dashed.

Proposition 7.14. *The relations $\beta_k \gamma_k = 0$ and $\gamma_k \alpha_k = 0$ hold in the formal contact category $Ho(Ko_+(\Sigma_{1,1}, 2))$.*

Proof. The logic is analogous to the proof of Proposition 7.6 above. The map $\beta_k \gamma_k$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass γ_k is capped implying that the composition $\beta_k \gamma_k$ factors through zero. This is illustrated below.

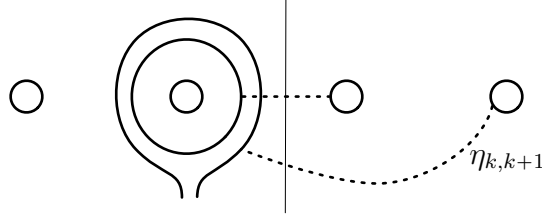


The map $\gamma_k \alpha_k$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass α_k is capped implying that the composition $\beta_k \gamma_k$ factors through zero. This illustrated below.



□

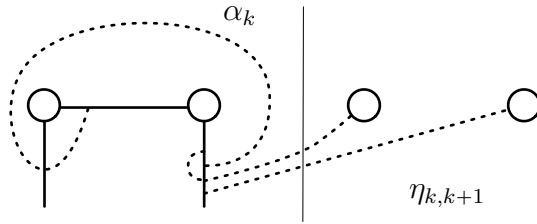
Step #2. As pictured above, the idempotents I_k correspond to the boundaries of regular neighborhoods of loops about the first 1-handle and the idempotents J_k to the boundaries of regular neighborhoods of loops about the second 1-handle in the k th torus $\Sigma_{1,1} \subset \Sigma_{g,1}$. The tori $\Sigma_{1,1}$ are ordered by the arc parameterization and, when two tori are adjacent, there is a bypass move $\eta_{k,k+1} : J_k \rightarrow I_{k+1}$ from the dividing set about the second 1-handle of the first torus to the dividing set about the first 1-handle of the second torus. The map $\eta_{k,k+1}$ is pictured below.



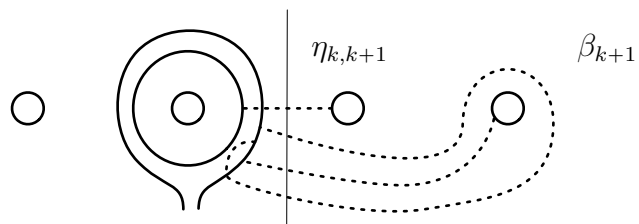
In the illustration above, the first two and the second two smaller circles are connected by annuli $S^1 \times [0, 1]$ to form the k th and $k + 1$ st tori $\Sigma_{1,1} \subset \Sigma_{g,1}$.

Proposition 7.15. *The relations: $\eta_{k,k+1}\alpha_k = 0 : I_k \rightarrow I_{k+1}$ and $\beta_k\eta_{k,k+1} = 0 : J_k \rightarrow J_{k+1}$ hold in the formal contact category $Ho(\mathcal{Ko}_+(\Sigma_{g,1}, 2))$*

Proof. The logic is analogous to the proof of Proposition 7.14 above. The map $\eta_{k,k+1}\alpha_k$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass $\eta_{k,k+1}$ is capped implying that the composition factors through zero. This is illustrated below.



The map $\beta_k\eta_{k,k+1}$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass β_k is capped implying that the composition factors through zero. This is illustrated below.



□

8. COMPARISON TO GEOMETRIC CATEGORIES

One of the appealing qualities of the formal contact category $\mathcal{Ko}(\Sigma)$ is that it has a universal property with respect to other dg categories by construction. Although there is no underlying Floer theory or contact geometry, this property allows us compare $\mathcal{Ko}(\Sigma)$ to other constructions which stem from observations involving the former or the latter. In this section we will discuss why the universal property of $\mathcal{Ko}(\Sigma)$ implies the existence of maps:

$$\begin{array}{ccc} & \mathcal{Ko}(\Sigma) & \\ \swarrow \text{dashed} & & \searrow \\ \mathcal{Co}(\Sigma) & & \mathcal{A}(-\mathcal{Z})\text{-mod} \end{array}$$

in the homotopy category of dg categories which relate contact categories $\mathcal{Co}(\Sigma)$ with the corresponding component of the Bordered Heegaard-Floer theory. See Sections 8.1 and 8.3 for precise statements.

8.1. Relation to the Contact Category. Much of the material in this paper was inspired by K. Honda's proposed *contact category* $\mathcal{Co}(\Sigma)$ [12]. Although a full account of this construction is in preparation, in this section a modest comparison is drawn between the formal and geometric contact categories.

The morphisms in the contact category $\mathcal{Co}(\Sigma)$ are tight contact structures on $\Sigma \times [0, 1]$. More precisely, $\mathcal{Co}(\Sigma)$ is the additivization [29, §1.1.2.1] of a category with objects given by dividing sets γ on the surface Σ and morphisms $\theta : \gamma \rightarrow \gamma'$ given by contactomorphism classes of contact structures on $\Sigma \times [0, 1]$, which induce γ and γ' on $\partial\Sigma \times [0, 1]$, subject to the relation that an overtwisted contact structure is zero. The composition is induced by the pullback of contact plane fields along the rescaling diffeomorphism: $\Sigma \times [0, 1] \xrightarrow{\sim} \Sigma \times [0, 1] \cup_{\Sigma} \Sigma \times [0, 1]$.

The contact category $\mathcal{Co}(\Sigma)$ plainly exists. The maps in the contact category $\mathcal{Co}(\Sigma)$ are generated by bypass moves between dividing sets [13, Lem. 3.10 (Isotopy discretization)]. Since the bypass moves satisfy the elementary relations (1) and (2) in Definition 3.16, there is a functor: $\sigma : \text{Pre-}\mathcal{Ko}(\Sigma) \rightarrow \mathcal{Co}(\Sigma)$. When (Σ, m) is a surface with boundary then the discussion in Section 4.9 suggests that these categories are very closely related.

For the purposes of comparison we must make the non-trivial assumption below.

Assumption. *The contact category $\mathcal{Co}(\Sigma)$ has pretriangulated dg enhancement $\mathcal{Co}^{\text{dg}}(\Sigma)$ in which bypass triangles are distinguished triangles.*

If this assumption is correct then there is a canonical lift

$$\tilde{\sigma} : \mathcal{Ko}(\Sigma) \rightarrow \mathcal{Co}^{\text{dg}}(\Sigma)$$

of the dg functor σ to a functor from the formal contact category to the dg category $\mathcal{Co}^{\text{dg}}(\Sigma)$.

Remark 8.2. In the formal contact category $\mathcal{Ko}(\Sigma)$, the bypass an involving the annulus in the proof of Theorem 4.18 determines a distinguished triangle:

$$\gamma \xrightarrow{an} \gamma' \xrightarrow{an'} \gamma'' \xrightarrow{an''} \gamma[1].$$

The map an' is not necessarily zero. However, in the geometric setting $an' = 0$, making the convolution $\gamma \simeq C(an')$ isomorphic to a direct sum [43]. As $\sigma(an') = 0$, it is possible to view $Ho(\mathcal{Ko}(\Sigma))$ as a deformation.

8.3. Relation to the Bordered Sutured Floer Categories. In this section a functor $\widetilde{\mathcal{Ko}}_+(\Sigma, m) \rightarrow \mathcal{A}(-\mathcal{Z})\text{-mod}$ from a cofibrant replacement of the positive part of the formal contact category to the category of left dg modules over an arc algebra of an arc diagram \mathcal{Z} that parameterizes Σ . Assume that (Σ, m) has at least one boundary component and every boundary component $\partial_i \Sigma$ contains a positive even number of points m_i . The ground ring k of $\mathcal{Ko}_+(\Sigma, m)$ is fixed to be the field \mathbb{F}_2 . We will not discuss gradings here. The cofibrant replacement is a slightly larger, but quasi-equivalent category, see Remark 3.19. In particular, there is a functor $\mathcal{Ko}_+(\Sigma, m) \rightarrow \mathcal{A}(-\mathcal{Z})\text{-mod}$ in Hqe.

If γ is a dividing set on Σ then R. Zarev associates a bordered sutured manifold [50, §3.2] called the *cap* W_γ to γ . The cap W_γ is the 3-manifold $\Sigma \times [0, 1]$ in which the surface $\Sigma \times \{0\}$ is parameterized by the arc diagram \mathcal{Z} , the *sutures* m are the m boundary points, the dividing set γ appears on $\Sigma \times \{1\}$ and the two sides are connected by straight lines segments in $\partial \Sigma \times [0, 1]$.

$$W_\gamma = (\Sigma \times [0, 1], \gamma \times \{1\} \cup \Lambda \times [0, 1], (-\Sigma \times \{0\}, -\Lambda \times \{0\})).$$

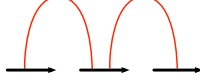
For details concerning this definition consult [51, Def. 2.5].

Associated to each bordered sutured manifold Y , there is a Heegaard diagram $H(Y)$ [50, §4]. Associated to each Heegaard diagram $H(Y)$ there is a left dg $\mathcal{A}(-\mathcal{Z})$ -module $\widehat{\text{BSD}}(Y)$ [50, §7.3]. Notation for the module does not include the intermediate Heegaard diagram because the homotopy type of the module is independent of this choice.

If γ is a dividing set on Σ such that the basepoint z_1 is contained in the positive region $R_+ \subset \Sigma \setminus \gamma$ then γ determines an object $\gamma \in \text{Ob}(\mathcal{Ko}_+(\Sigma, m))$. To each such γ we associate the left dg module $\widehat{\text{BSD}}(\gamma) = \widehat{\text{BSD}}(W_\gamma)$ associated to the cap for some choice of Heegaard diagram.

$$\gamma \mapsto \widehat{\text{BSD}}(\gamma) \quad \text{where} \quad \widehat{\text{BSD}}(\gamma) = \widehat{\text{BSD}}(W_\gamma) \quad (8.1)$$

The disk $(D^2, 6)$ can be parameterized by an arc diagram \mathcal{W}_3 pictured below:



The diagram \mathcal{W}_3 consists of three oriented line segments $Z = \{Z_1, Z_2, Z_3\}$ containing the points $\{a\}$, $\{a' < b\}$ and $\{b'\}$ respectively. The matching function M is determined by $M(a) = M(a')$ and $M(b) = M(b')$.

As discussed in Proposition 3.17, the three important dividing sets γ_A , γ_B and γ_C in $(D^2, 6)$ can be connected three bypass moves

$$\gamma_A \xrightarrow{\theta_A} \gamma_B \xrightarrow{\theta_B} \gamma_C \xrightarrow{\theta_C} \gamma_C[1] \quad \text{or} \quad \overline{\overline{\quad}} \xrightarrow{\theta_A} \succ \searrow \xrightarrow{\theta_B} \succ \searrow \xrightarrow{\theta_C} \overline{\overline{\quad}} [1].$$

(The signs of the regions are fixed by requiring that the region containing the basepoint is positive.) Associated these three dividing sets, there are three left $\mathcal{A}(-\mathcal{W}_3)$ -modules $\widehat{\text{BSD}}(\gamma_A)$, $\widehat{\text{BSD}}(\gamma_B)$ and $\widehat{\text{BSD}}(\gamma_C)$ corresponding to the bordered sutured diagrams given by the caps W_{γ_A} , W_{γ_B} and W_{γ_C} .

In [9, §6.2], the authors J. B. Etnyre, D. S. Vela-Vick and R. Zarev made a fundamental computation: after choosing Heegaard diagrams for the caps W_{γ_A} , W_{γ_B} and W_{γ_C} , they find that there are chain maps: $\phi_A : \widehat{\text{BSD}}(\gamma_A) \rightarrow \widehat{\text{BSD}}(\gamma_B)$, $\phi_B : \widehat{\text{BSD}}(\gamma_B) \rightarrow \widehat{\text{BSD}}(\gamma_C)$ and $\phi_C : \widehat{\text{BSD}}(\gamma_C) \rightarrow \widehat{\text{BSD}}(\gamma_A)$ such that

$$\widehat{\text{BSD}}(\gamma_A) \xrightarrow{\phi_A} \widehat{\text{BSD}}(\gamma_B) \xrightarrow{\phi_B} \widehat{\text{BSD}}(\gamma_C) \xrightarrow{\phi_C} \widehat{\text{BSD}}(\gamma_A)[1]$$

is a distinguished triangle. They show explicitly that

- (1) $\widehat{\text{BSD}}(\gamma_A) = C(\phi_B)$
- (2) ϕ_A is projection and
- (3) ϕ_C is inclusion.

(Alternatively, this follows from Section 6 and the Morita invariance of the category associated to the disk by Bordered Sutured Floer theory.) Our functor is defined using the pairing theorem to extend the assignments: $\theta_A \mapsto \phi_A$, $\theta_B \mapsto \phi_B$ and $\theta_C \mapsto \phi_C$ to all of the other bypass moves between dividing sets.

Throughout the remainder of this section, we will make repeated use the pairing theorem. Suppose that γ is a dividing set on Σ and the first basepoint z_1 is contained in a positive region. Then if $D = (D^2, 2m) \subset \Sigma$ is an embedded disk with $2m$ points on the boundary such that $\gamma^\circ = \gamma \setminus (D \cap \gamma)$ is a dividing set on $\Sigma \setminus D$ then the pairing theorem [50, Thm. 8.7] gives a homotopy equivalence:

$$\widehat{\text{BSD}}(\gamma) \xrightarrow{\sim} \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma \cap D) \quad \gamma = \gamma^\circ \cup_{\gamma \cap \partial D} (\gamma \cap D)$$

where $\gamma \cap \partial D = 2m$, $\widehat{\text{BSD}}(\gamma) = \widehat{\text{BSD}}(W_\gamma)$ is the left dg $\mathcal{A}(-\mathcal{Z})$ -module assigned to the dividing set γ , $\widehat{\text{BSDA}}(\gamma^\circ) = \widehat{\text{BSDA}}(W_{\gamma^\circ})$ is a left $\mathcal{A}(-\mathcal{Z})$ -module and right A_∞

$\mathcal{A}(-\mathcal{W}_3)$ -module, $\widehat{\text{BSD}}(\gamma \cap D) = \widehat{\text{BSD}}(W_{\gamma \cap D})$ is the left $\mathcal{A}(-\mathcal{W}_3)$ -module determined by γ in the interior of the disk D and the box product \boxtimes is an analogue of the derived tensor product, see [26, §2.4].

Definition 8.4. If $\theta : \gamma \rightarrow \eta$ is a bypass move then the map $\theta_* : \widehat{\text{BSD}}(\gamma) \rightarrow \widehat{\text{BSD}}(\eta)$ of dg modules associated to θ is determined by the commutative diagram:

$$\begin{array}{ccc} \widehat{\text{BSD}}(\gamma) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma_A) \\ \downarrow \theta_* & & \downarrow 1 \boxtimes \phi_A \\ \widehat{\text{BSD}}(\eta) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma_B) \end{array}$$

where $\gamma^\circ = \gamma \setminus D$, introduced above, denotes the dividing set minus the region containing the equator of the bypass disk associated to θ .

In order for the maps chosen above to yield a functor from the pre-formal contact category, we must check that relations (1) and (2) in Definition 3.16 above are satisfied. Since these relations hold up to homotopy in the category $\mathcal{A}(-\mathcal{Z})\text{-mod}$, this determines a functor from the cofibrant replacement of the pre-formal contact category. Lastly we will show that this functor factors through the Postnikov localization introduced by Proposition 3.17.

Relation (1). If θ is capped in the northwest or southeast then relation (1) must hold up to homotopy by the invariance of the bordered sutured theory [50, §7].

In more detail, suppose that $\theta : \gamma \rightarrow \eta$ is a bypass move and D is a neighborhood of the equator of the underlying bypass disk. Then when there is a cap, the region D can be enlarged to a region \tilde{D} which contains the cap disk in Σ . Two applications of the pairing theorem give:

$$\begin{array}{ccccccc} \widehat{\text{BSD}}(\gamma) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma_A) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\tilde{\gamma}^\circ) \boxtimes \widehat{\text{BSD}}(\tilde{\gamma}_A) \\ \downarrow \theta_* & & \downarrow 1 \boxtimes \phi_A & & \downarrow 1 \boxtimes \tilde{\phi}_A \\ \widehat{\text{BSD}}(\eta) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma_B) & \xrightarrow{\sim} & \widehat{\text{BSDA}}(\tilde{\gamma}^\circ) \boxtimes \widehat{\text{BSD}}(\tilde{\gamma}_B) \end{array}$$

where $\gamma^\circ = \gamma \setminus D$ and $\tilde{\gamma}^\circ = \gamma \setminus \tilde{D}$. The dividing sets $\tilde{\gamma}_A$ and $\tilde{\gamma}_B$, on the righthand side above, are identical when the cap is either northwestern or southeastern. They are both represented by the same Heegaard diagram and the map $\tilde{\phi}_A$ is identity. It follows that θ_* is homotopic to identity.

Relation (2). In order to see that disjoint bypass moves $\theta \sqcup \theta' : \gamma \rightarrow \eta$ commute we must cut the dividing set γ along the two disjointly embedded disks corresponding to neighborhoods of the equators of our bypass moves to form $\gamma^{\circ\circ} = \gamma \setminus (D \sqcup D')$. The arc algebra associated to a disjoint union splits, $\varphi : \mathcal{A}(-(\mathcal{W}_3 \sqcup \mathcal{W}_3)) \xrightarrow{\sim} \mathcal{A}(-\mathcal{W}_3) \otimes_k \mathcal{A}(-\mathcal{W}_3)$, the module $\widehat{\text{BSD}}(\gamma_A) \otimes_k \widehat{\text{BSD}}(\gamma_A)$ appears in the pairing theorem:

$$\widehat{\text{BSD}}(\gamma) \xrightarrow{\sim} \widehat{\text{BSDA}}(\gamma^{\circ\circ}) \boxtimes \left[\widehat{\text{BSD}}(\gamma_A) \otimes_k \widehat{\text{BSD}}(\gamma_A) \right]$$

and the disjoint union of Heegaard diagrams splits as a tensor product compatible with the isomorphism φ above. Under this identification, the maps θ_* and θ'_* induced by θ and θ' correspond to different tensor factors and must commute by the standard algebraic fact that:

$$(1_{\gamma^{\circ\circ}} \boxtimes [1_A \otimes \theta'_*])(1_{\gamma^{\circ\circ}} \boxtimes [\theta_* \otimes 1_A]) = (1_{\gamma^{\circ\circ}} \boxtimes [\theta_* \otimes 1_A])(1_{\gamma^{\circ\circ}} \boxtimes [1_A \otimes \theta'_*])$$

where $1_{\gamma^{\circ\circ}}$ and 1_A are used to denote the identity maps $1_{\widehat{\text{BSDA}}(\gamma^{\circ\circ})}$ and $1_{\widehat{\text{BSD}}(\gamma_A)}$ respectively.

Triangles. Finally, it is necessary to see that the objects and the maps assigned by Equation (8.1) and Definition 8.4 factor through the Postnikov localization constructed in Proposition 3.17

These choices form distinguished triangles because

$$\begin{aligned} \widehat{\text{BSD}}(\gamma) &= \widehat{\text{BSD}}(\gamma \cup \gamma_A) \\ &\simeq \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes \widehat{\text{BSD}}(\gamma_A) \\ &\simeq \widehat{\text{BSDA}}(\gamma^\circ) \boxtimes C(\phi_B) \\ &\simeq C(1_{\widehat{\text{BSDA}}(\gamma^\circ)} \boxtimes \phi_B) \\ &\simeq C(\theta'_*) \end{aligned}$$

where the last equivalence corresponds to the commutative diagram in Definition 8.4 above after rotating the triangle. An analogue of this argument appears in [27, Thm. 4.1].

Acknowledgments. The construction of contact categories was inspired by the ideas of K. Honda, Y. Tian and K. Walker [12, 41, 42, 47]. The author would especially like to thank Y. Tian for his helpful correspondence and the Simons Center for facilitating our discussion. Also Y. Huang, R. Lipshitz and I. Petkova for several helpful emails. My colleagues C. Frohman, A. Kaloti and R. Kinser for their cordiality and their conversation. This project began after several conversations with A. Kaloti.

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9. GLOSSARY OF NOTATION

After Section 2 dg categories are ungraded over a field of characteristic 2. The homotopy category of dg categories $Ho(dgcat_k)$ over k will be denoted by Hqe or Hmo when the equivalence relation is quasi-equivalence or Morita equivalence respectively. All surfaces denoted by Σ are connected unless otherwise mentioned. $\Sigma_{g,n}$ is the orientable surface of genus g with n boundary components.

$-^\vee$	Prop. 4.7.	$\gamma_A, \gamma_B, \gamma_C$	bypass triangle, Prop. 3.17, §8.3.
$-^{op}$	opposite category.		
\mathbf{a}	points $\{a_1, \dots, a_{2k}\}$ in arc diagram, Def. 5.6.	γ^\vee	dual dividing set, Def. 3.5, Prop. 4.7.
a_k, a'_k	points in an arc diagram, Def. 5.6.	$(\oplus_{i=1}^n \gamma_i, p)$	convolution of dividing sets, Def. 2.2.
$\mathcal{A}(\mathcal{Z})$	arc algebra, [26, 50].		
B	bottom of D^2 .	$\Gamma(\Sigma)$	mapping class group, §5.1.
\widehat{B}	$B \subset \Sigma$, Def. 4.17.	h_k	1-handle in $F(\mathcal{M}_n)$.
$\widehat{BSD}(\gamma)$	Eqn. 8.1.	$Ho(\mathcal{C})$	$[\mathcal{C}]$ or $H^0(\mathcal{C})$, [44].
\mathcal{C}	dg category, After §2 ungraded, see §2.25.	Hom^I	Def. 2.8.
c_i	cocore of 1-handle.	Hom^T	Def. 2.18.
$\mathcal{Co}(\Sigma)$	geometric contact category or Y. Tian algebraic contact category.	$Hom^{\langle K \rangle}$	Prop. 2.23.
d	differential, $d^2 = 0$.	Hmo	Morita homotopy category, [37].
$dgcat_k$	category of dg categories, [6, 44].	Hqe	homotopy category, [39].
D', \bar{D}, \tilde{D}	Def. 2.12, Def. 2.16.	$i(x, y)$	geometric intersection number.
D^2	unit disk.	$int(X)$	interior of X .
e_k	generator of \mathcal{N}_n , Def. 6.8.	I, I', \bar{I}	Def. 2.7 and Def. 2.9.
$\mathfrak{e}(\gamma)$	Def. 4.2.	k	ground field. After §2, $char(k) = 2$.
$F(\mathcal{Z})$	surface of arc diagram, Def. 5.7.	κ, κ'	Def. 2.7 and Def. 2.16.
γ	dividing set, Def. 3.4.	$\langle K \rangle$	Prop. 2.23.
$\gamma(\epsilon_i)$	dividing set associated to e_i , §6.7.	$K_0(\mathcal{C})$	Grothendieck group, [37].
		$\mathcal{Ko}(\Sigma)$	Def. 3.18.

$\mathcal{Ko}^n(\Sigma, m)$	Thm. 4.5.	$\Sigma_{g,n}$	connected surface of genus g with n boundary components.
$\mathcal{Ko}_{\pm}^n(\Sigma, m)$	\pm -halves of $\mathcal{Ko}^n(\Sigma, m)$, §4.19.	(Σ, m)	pointed oriented surface, Def. 3.2.
$L_R\mathcal{C}$	Def. 2.7.	$\bar{\Sigma}$	orientation reversal, Prop. 4.8.
$L_S\mathcal{C}$	Def. 2.19.	$\partial_i\Sigma$	i th boundary component of Σ , Def. 3.2.
m	boundary points $m \subset \partial\Sigma$, §3.1.	$\theta : \gamma \rightarrow \gamma'$	bypass move, Def. 3.9.
M	a matching $M : \mathbf{a} \rightarrow \{1, \dots, k\}$ in arc diagram, Def. 5.6.	$\theta_{i,j}$	Def. 2.12, Def. 2.16.
μ	§6.14.	T	top of D^2 .
\mathcal{M}_n	zig-zag diagram for $(D^2, 2n)$, Def. 6.2.	(T, γ, γ')	bypass attachment, Def. 3.9.
$\text{Mat}(\mathcal{C})$	the additive closure, §2.1.	W_γ	cap associated to γ , [51, Def. 2.5], §8.3.
\mathcal{N}_n	nil-Temperley-Lieb algebra, Def. 6.8.	Ξ	Eqn. (3.15), Def. 3.18.
\mathbb{N}	$\mathbb{N} = \{0\} \cup \mathbb{Z}_+$.	\mathcal{Y}_n	$\mathcal{R}_n^{\text{pretr}}$, Def. 6.13.
nS^1	Def. 4.11.	z	basepoints $z = \{z_1, \dots, z_n\}$, $z_i \in \partial_i\Sigma$, Def. 3.2.
$N(T)$	neighborhood of disk, Def. 3.9.	\mathfrak{z}_C	$z_C \in \mathfrak{Z}(\mathcal{Z})$, Def. 5.12.
$\text{Pre-}\mathcal{Ko}(\Sigma)$	Def. 3.16.	\mathbb{Z}_+	$\{1, 2, 3, \dots\} \subset \mathbb{Z}$.
$\text{Pre-Pre-}\mathcal{Ko}(\Sigma)$	Rmk. 3.19.	$\mathbb{Z}/2$	$\mathbb{Z}/2\mathbb{Z}$.
\mathcal{Q}_n	Y. Tian quiver, Def. 6.9.	Z	ordered set of lines, Def. 5.6.
r	Basepoint automorphism, Cor 5.4.	\mathcal{Z}	arc diagram, Def. 5.6.
$\rho_{k,k\pm 1}$	Eqn. (6.3).	\mathcal{Z}_i	arc in arc diagram, Def. 5.6.
\mathcal{R}_n	Y. Tian disk category, Def. 6.12.	$\mathcal{Z}_{0,n}$	arc diagram for $\Sigma_{0,n}$, Def. 7.2.
R_{\pm}	positive and negative regions, Def. 3.4.	$\mathcal{Z}_{g,1}$	arc diagram for $\Sigma_{g,1}$, Def. 7.10.
S^2	the 2-sphere.	$\mathfrak{Z}(\mathcal{Z})$	set of elementary dividing sets, Def. 5.12.

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